

EUCLIDE'S ELEMENTS;

The whole FIFTEEN BOOKS
compendiously Demonstrated :

WITH

ARCHIMEDES'S Theorems
of the Sphere and Cylinder Investi-
gated by the *Method of Indivisibles*.

By ISAAC BARROW, D.D. *late Ma-
ster of Trinity College in Cambridge.*

To which is ANNEX'D

EUCLIDE'S *Data with Ma-
rinus's Preface, and a brief
Treatise of Regular Solids.*

And in this EDITION is added

A SUPPLEMENT, contain-
ing some Practical Corollaries dedu-
ced from some of the most material
Propositions in EUCLIDE,

Καθαροὶ λογικῆς ἀρχαὶ εἰσὶν αἱ μαθηματικαὶ ἐπιστήμαι

L O N D O N, Printed by J. REDMAYNE,
for J. and B. SPRINT, and A. WARD in
Little-Britain, and T. PAGE and W. MOUNT
on *Tower-Hill*. 1722.



To the READER.

IF you are desirous, Courteous Reader, to know what I have perform'd in this Edition of the Elements of Euclide, I shall here explain it to you in short, according to the nature of the Work. I have endeavour'd to attain two ends chiefly; the first, to be very perspicuous, and at the same time so very brief, that the Book may not swell to such a Bulk, as may be troublesome to carry about one, in both which I think I have succeeded. Some of a brighter Genius, and endued with greater Skill, may have demonstrated most of these Propositions with more nicety, but perhaps none with more succinctness than I have; especially since I alter'd nothing in the number and order of the Author's Propositions; nor presum'd either to take the liberty of rejecting, as less necessary, any of them, or of reducing some of the easier sort into the rank of Axioms, as several have done; and among others, that most expert Geometrician A. Tacquetus C. (whom I the more willingly name, because I think it is but civil to acknowledge that I have imitated him in some Points) after whose most accurate Edition I had no Thoughts of attempting any thing of this nature, till I consider'd that this most learned Man thought fit to publish only eight of Euclide's Books, which he took the pains to explain and embellish, having in a manner rejected and undervalued the other seven, as less appertaining to the Elements of Geometry. But my Province was

To the R E A D E R.

originally quite different, not that of writing the Elements of Geometry after what method sever I pleas'd, but of demonstrating, in as few words as possible I cou'd, the whole Works of Euclide. As to four of the Books, viz. the seventh, eighth, ninth, and tenth, altho' they don't so nearly appertain to the Elements of plain and solid Geometry, as the six precedent and the two subsequent, yet none of the more skilful Geometricians can be so ignorant as not to know that they are very useful for Geometrical matters, not only by reason of the mighty near affinity that is between Arithmetick and Geometry, but also for the knowledge of both measurable and unmeasurable Magnitudes, so exceeding necessary for the Doctrine of both plain and solid figures. Now the noble Contemplation of the five Regular Bodies that is contain'd in the three last Books, cannot without great Injustice be pretermitted, since that for the sake thereof our *συγγραφεύς*, being a Philosopher of the Platonic Sect, is said to have compos'd this universal System of Elements; as Proclus lib. 2. witnesseth in these Words, *Ὁθεν δὴ καὶ τῆς συμπάντος συγγραφῆς τέλος προεβίβαστο τὴν τῷ καλεμῶν πλατωνικῶν ἀναγνώσκοντος.* Besides, I easily perswaded my self to think, that it would not be unacceptable to any Lover of these Sciences to have in his possession the whole Euclidean Work, as it is commonly cited and celebrated by all Men: Wherefore I resolv'd to omit no Book or Proposition of those that are found in P Herigonius's Edition, whose Steps I was oblig'd closely to follow, by reason I took a Resolution to make use of most of the Schemes of the said Book, very well foreseeing that time would not allow me to form

TO the READER.

new ones, tho' sometimes I chose rather to do it. For the same reason I was willing to use for the most part Euclide's own Demonstrations, having only express'd them in a more succinct Form, unless perhaps in the second, thirteenth, and very few in the seventh, eighth, and ninth Books, in which it seem'd not worth my while to deviate in any particular from him: Therefore I am not without good hopes that as to this part I have in some measure satisfied both my own Intentions, and the Desire of the Studious. As for some certain Problems and Theorems that are added in the Scholions (or short Expositions) either appertaining (by reason of their frequent Use) to the nature of these Elements, or conducing to the ready Demonstration of those things that follow, or which do intimate the reasons of some principal Rules of practical Geometry, reducing them to their original Fountains, these I say, will not, I hope, make the Book swell to a Size beyond the design'd Proportion.

The other Butt which I levell'd at, is to content the Desires of those who are delighted more with symbolical than verbal Demonstrations. In which kind, whereas most among us are accusom'd to the Symbols of Gulielmus Oughtredus, I therefore thought best to make use, for the most part, of his. None hitherto (as I know of) has attempted to interpret and publish Euclide after this manner, except P. Herigonius; whose Method (tho' indeed most excellent in many things, and very well accommodated for the particular purpose of that most ingenious Man) yet seems in my Opinion to labour under a double Defect. First, in regard that, altho' of two or more Propositions produ-
ced

TO the READER.

ced for the Proof of any one Problem or Theorem, the former don't always depend of the latter, yet it don't readily enough appear, either from the order of each or by any other manner, when they agree together, and when not; wherefore for want of the Conjunctions and Adjectives, ergo, rursus, &c. many difficulties and occasions of doubt do often arise in reading, especially to those that are Novices. Besides it frequently happens, that the said Method cannot avoid superfluous Repetitions, by which the Demonstrations are oftentimes render'd tedious, and sometimes also more intricate; which Faults my Method doth easily remedy by the arbitrary mixture of both Words and Signs: Therefore let what has been said, touching the Intention and Method of this little Work, suffice. As to the rest, whoever covets to please himself with what may be said, either in Praise of the Mathematicks in general, or of Geometry in particular, or touching the History of these Sciences, and consequently of Euclide himself, (who digested those Elements) and others *ἐκτετακται* of that kind, may consult other Interpreters. Neither will I (as if I were afraid lest these my Endeavours may fall short of being satisfactory to all Persons) alledge as an Excuse (tho' I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occasion'd by other Affairs, nor yet the want of requisite help for these Studies, nor several other things of the like nature. But what I have here employ'd my Labour and Study in for the Use of the ingenious Reader, I wholly submit to his Censure and Judgment, to approve if useful, or reject if otherwise.

I. B.

Ad amicissimum Virum, I. C. de EUCLIDE
contracto, Εὐκλείστους.

Factum bene! didicit Laconice loqui
Senex profundus, & aphorismos induit.
Immensa dudum margo commentarii
Diagramma circuit minutum; utque Insula
Problema breve natabat in vasto mari.
Sed unda jam detumuit; & glossa arctior
Stringit Theoremata: minoris anguli
Lateribus ecce totus Euclides jacet,
Inclusus olim velut Homerus in nuce;
Pluteoque sarcina modo qui incubuit, levis
En fit manipulus. Pelle in exigua latet
Ingens Mathesis, matris utero Hercules,
In glande quercus, vel Ithaca Eurys in pila.
Nec mole dum decrescit, usu fit minor;
Quin auctior jam evadit, & cumulatus
Contracta prodest erudita pagina.
Sic ubere magis liquor è presso affluit;
Sic pleniori vasa inundat sanguinis
Torrente cordis Systole; sic fusius
Procurrit aquor ex Abyla angustis.
Tantilli operis ars tanta referenda unice est
BAROVIANO nomini, ac solertia.
Sublimis euge mentis ingenium potens!
Cui invium nil, arduum esse nil solet;
Sic usque pergas prospero conamine,
Radiusque multum debeat ac abacus tibi;
Sic crescat indies feracior seges,
Simili colonum germine assiduo beans.
Specimen futuri messis hic fiet labor.
Magnaue fama illustra hæc præludia.
Juvenis dedit qui tanta, quid dabit senex?

Car. Robotham, CANTAB.
Coll. Trin. Sen. Sec.

The

The Explication of the Signs or Characters.

$\equiv$	Signifies	Equal.
$>$		Greater.
$<$		Lesser.
$+$		More, or to be added.
$-$		Less, or to be subtracted.
$\vdash$		The Differences, or Excess; Also, that all the quantities which follow, are to be subtracted, the Signs not being changed.
$\times$		Multiplication, or the Drawing one side of a Rectangle into another. The same is denoted by the Conjunction of letters; as $AB = A \times B$.
$::$		Continued Proportion.
$\sqrt{\quad}$		The Side or Root of a Square, or Cube, &c
$Q \& q$		A Square.
$C \& c$		A Cube.
$Q. Q.$		The ratio of a square number to a square number.

Other Abbreviations of words, where-ever they occur, the Reader will without trouble understand of himself; saving some few, which, being of less general use, we refer to be explained in their places, most commonly at the beginning of each Book in which they are used.

The



The FIRST BOOK
OF
EUCLIDE'S
ELEMENTS.

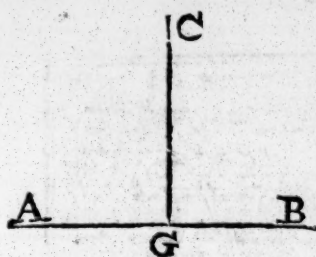
Definitions.

- I.  Point is that which hath no part.
 II. A Line is a longitude without latitude.
 III. The ends, or limits, of a line are points.
 IV. A right line is that which lies equally betwixt it's points.
 V. A Superficies is that which hath only longitude and latitude.
 VI. The extremes, or limits, of a superficies are lines.
 VII. A plain superficies is that which lies equally betwixt it's lines.
 VIII. A plain angle is the inclination of two lines the one to the other, the one touching the other in the same plain, yet not lying in the same strait line.
 IX. And if the lines, which contain the angle, be right lines, it is called a right-lined angle.

A

X. When

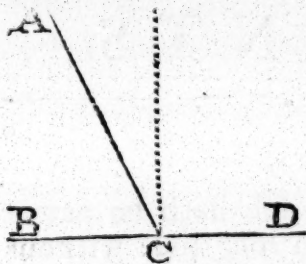
The first Book of



X. When a right line CG, standing upon a right line AB, makes the angles on either side thereof, CGA, CGB, equal one to the other, then both those equal angles are right angles; and the right line

CG, which standeth on the other, is termed a Perpendicular to that (AB) whereon it standeth.

Note, When several angles meet at the same point (as at G) each particular angle is described by three letters; whereof the middle letter sheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right lines CG, AG make at G, is called CGA, or AGC.



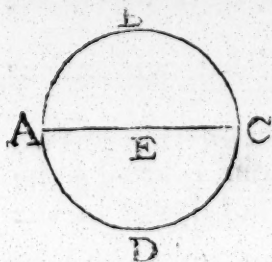
XI. An obtuse angle is that which is greater than a right angle; as ACD.

XII. An acute angle is that which is less than a right angle; as ACB.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

XV. A Circle is a plain figure contained under one line, which is called a Circumference; unto which all lines, drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.



XVI. And that point is called the Center of the circle.

XVII. A Diameter of a circle is a right line drawn thro' the center thereof, and ending at the circumference on either side, dividing the

circle into two equal parts.

XVIII. A

XVIII. A Semicircle is a figure which is contained under the diameter and that part of the circumference which is cut off by the diameter.

In the circle EABCD, E is the center, AC the diameter, ABC the semicircle.

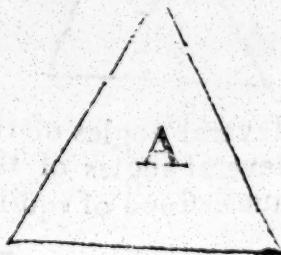
XIX. Right-lined figures are such as are contained under right lines.

XX. Three-sided or trilateral figures are such as are contained under three right lines.

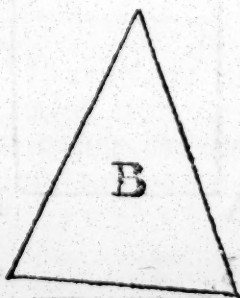
XXI. Four-sided or quadrilateral figures are such as are contained under four right lines.

XXII. Many-sided figures are such as are contained under more right lines than four.

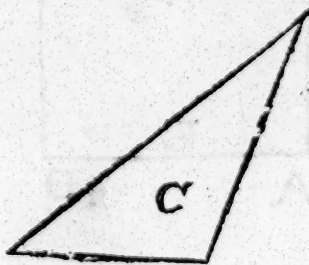
XXIII. Of trilateral figures, that is, an equilateral triangle, which hath three equal sides; as the triangle A.



XXIV. Isosceles is a triangle which hath only two sides equal; as the triangle B.

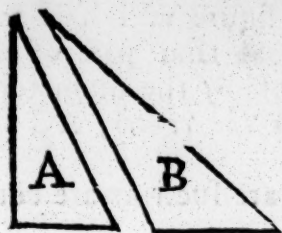


XXV. Scalenum is a triangle whose three sides are all unequal; as C.



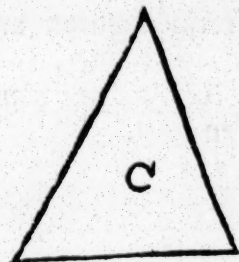
A 2

XXVI. Of



XXVI. Of these trilateral figures, a right-angled triangle is that which hath one right angle; as the triangle A.

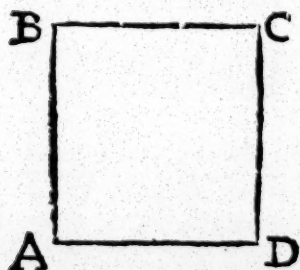
XXVII. An amblygonium, or obtuse-angled triangle, is that which hath one angle obtuse; as B.



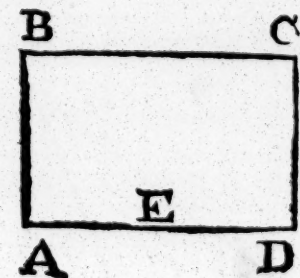
XXVIII. An oxygonium, or acute-angled triangle, is that which hath three acute angles; as C.

An equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the

several angles of the one figure be equal to the several angles of the other. The same is to be understood of equilateral figures.



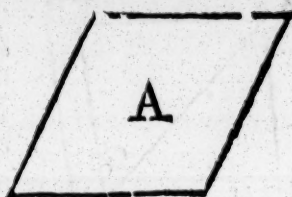
XXIX. Of quadrilateral, or four-sided figures, a square is that whose sides are equal, and angles right; as ABCD.



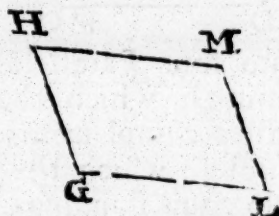
XXX. A Figure on the one part longer, or a long square, is that which hath right angles, but not equal sides; as ABCD.

XXXI. A

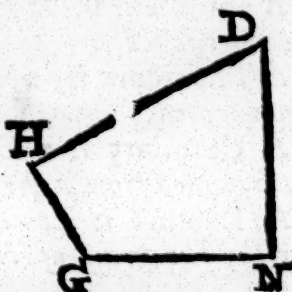
XXXI. A Rhombus, or Diamond-figure, is that which has four equal sides, but is not right-angled; as A.



XXXII. A Rhomboides, is that whose opposite sides, and opposite angles, are equal; but has neither equal nor right angles; as GLMH.

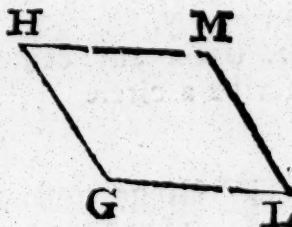


XXXIII. All other quadrilateral figures besides these are called *trapezia*, or tables; as GNDH.



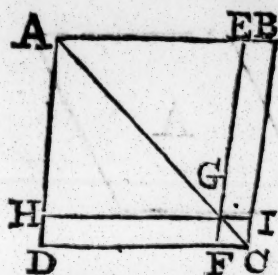
A ——— XXXIV. Parallel, or equidistant right lines are such, which being in the same superficies, if infinitely produced, would never meet; as A and B.

XXXV. A Parallelogram is a quadrilateral figure, whose opposite sides are parallel, or equidistant; as GLHM.



A ;

XXXVI. In



XXXVI. In a Parallelogram ABCD, when a diameter AC, and two lines EF, HI, parallel to the sides, cutting the diameter in one and the same point G, are drawn, so that the Parallelogram be divided by them into four Parallelograms; those two DG, GB, through which the diameter passeth not, are called complements; and the other two HE, FI, through which the diameter passeth, the Parallelograms standing about the diameter.

A Problem is, when something is proposed to be done or effected.

A Theoreme is, when something is proposed to be demonstrated.

A Corollary is a consequent, or some consequent truth gained from a preceding demonstration.

A Lemma is the demonstration of some premise, whereby the proof of the thing in hand becomes the shorter.

Postulates or Petitions.

1. **F**rom any given point to any other given point to draw a right line.
2. To produce a right line finite, strait forth continually.
3. Upon any center, and at any distance, to describe a circle

Axioms.

1. **T**Hings equal to the same thing, are also equal one to the other.

As $A=B=C$. Therefore $A=C$; or therefore all, A, B, C, are equal the one to the other.

Note, When several quantities are joined the one to the other continually with this mark $=$, the first quantity is by virtue of this axiom equal to the last, and every one to every one; In which case we often abstain

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abstain from citing the axiom, for brevity's sake; altho' the force of the consequence depend thereon.

2. If to equal things you add equal things, the wholes shall be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the same third, or to equal things, are equal one to the other. Understand the same of triple, quadruple, &c.

7. Things which are half of one and the same thing, or of things equal, are equal the one to the other. Conceive the same of subtriple, subquadruple, &c.

8. Things which agree together, are equal one to the other.

The converse of this axiom is true in right lines and angles, but not in figures, unless they be like.

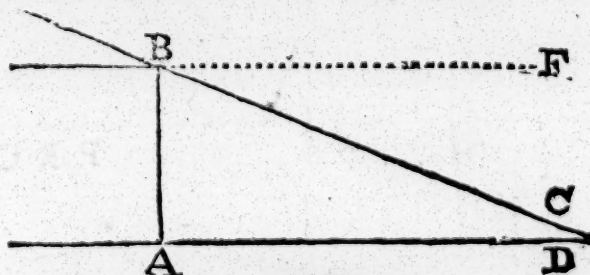
Moreover, magnitudes are said to agree, when the parts of the one being apply'd to the parts of the other, they fill up an equal or the same place.

9. Every whole is greater than it's part.

10. Two right lines cannot have one and the same segment (or part) common to them both.

11. Two right lines meeting in the same point, if they be both produced, they shall necessarily cut one the other in that point.

12. All right angles are equal the one to the other.



13. If a right line BA, falling on two right lines
A 4
AD, CB,

The first Book of

AD, CB, make the internal angles on the same side, BAD, ABC, less than two right angles, those two right lines produced shall meet on that side, where the angles are less than two right angles.

14. Two right lines do not contain a space.

15. If to equal things you add things unequal, the excess of the wholes shall be equal to the excess of the additions.

16. If to unequal things equal be added, the excess of the wholes shall be equal to the excess of those which were at first.

17. If from equal things unequal things be taken away, the excess of the remainders shall be equal to the excess of what was taken away.

18. If from things unequal things equal be taken away, the excess of the remainders shall be equal to the excess of the wholes.

19. Every whole is equal to all its parts taken together

20. If one whole be double to another, and that which is taken away from the first be double to that which is taken away from the second, the remainder of the first shall be double to the remainder of the second.

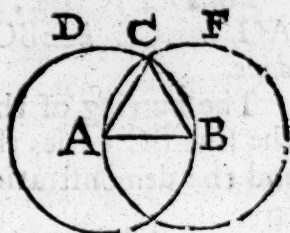
The Citations are to be understood in this manner; When you meet with two numbers, the first shews the Proposition, the second the Book; as by 4. 1. you are to understand the fourth Proposition of the first Book; and so of the rest. Moreover, ax. denotes Axiom, post. Postulate, def. Definition, sch. Scholium, cor. Corollary.

PROPOSITION I.

UPON a finite right line given AB, to describe an equilateral triangle ACB.

From the centers A and B, at the distance of AB, or BA, describe two circles cutting each other in the point C; from whence

draw two right lines CA, CB. Then is AC = AB = BC = AC. Wherefore the triangle ACB is equilateral. Which was to be done.



a 3. post.

b 1. post.

c 15. def.

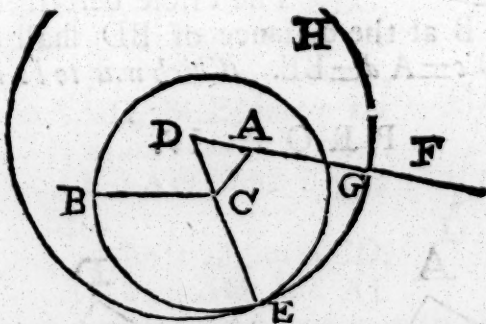
d 1. ax.

e 23. def.

Scholium.

After the same manner upon the line AB may be described an Isosceles triangle, if the distances of the equal circles be taken greater or less than the line AB.

PROP. II.



At a point given A, to make a right line AG equal to a right line given BC.

From the center C, at the distance of CB, describe the circle CBE. Join AC; upon which raise the equilateral triangle ADC. Produce DC to E. From the center D, at the distance of DE, describe the circle DEH. From the center A, at the distance of AC, describe the circle AFG. The circles DEH and AFG intersect at point G. The line segment AG is the required line equal to BC.

de-

e 2. *post.* describe the circle DEH; and let DA *e* be produced to the point G in the circumference thereof. Then $AG = CB$.

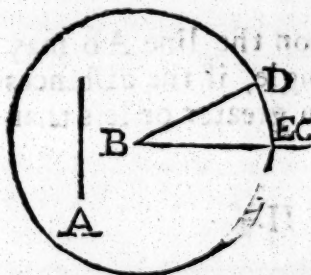
f 15. *def.* For $DG = DE$, and $DA = DC$. Wherefore
g *constr.* $AG = CE$ $k = BC$ $l = AG$. Which was to be
h 3. *ax.* done.

k 15. *def.* The putting of the point A within or without
l 1. *ax.* the line BC varies the cases; but the construction, and the demonstration, are every where alike.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no postulate, as Proclus well intimates.

P R O P. III.



Two right lines, A and BC, being given, from the greater BC to take away the right line BE equal to the lesser A.

a 2. *i.*

At the point B draw the right line $BD = A$.

b 15. *def.*

c *constr.*

d 1. *ax.*

The circle described from the center B at the distance of BD shall cut off $BE = BD$ $c = A$ $d = BE$. Which was to be done.

P R O P. IV.



If two triangles BAC, EDF, have two sides of the one BA, AC, equal to two sides of the other ED, DF, each to it's correspondent side (that is, $BA = ED$, and

and $AC=DF$) and have the angle A equal to the angle D contained under the equal right lines; they shall have the base BC equal to the base EF ; and the triangle BAC shall be equal to the triangle EDF ; and the remaining angles B, C , shall be equal to the remaining angles E, F , each to each, under which the equal sides are subtended.

If the point D be apply'd to the point A , and the right line DE plac'd upon the right line AB , the point E shall fall upon B , because $DE = AB$, *a hyp.* also the right line DF shall fall upon AC , because the angle $A = D$. Moreover the point F shall fall upon the point C , because $AC = DF$. Therefore the right lines EF, BC , shall agree, because they have the same terms, and consequently are equal. Wherefore the triangles, BAC, DEF , and the angles B, E , as also the angles C, F , do agree, and are equal. Which was to be demonstrated.

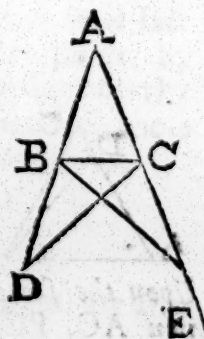
P R O P. V.

The angles ABC, ACB , at the base of an Isosceles triangle ABC , are equal one to the other: And if the equal sides AB, AC , be produced, the angles CBD, BCE , under the base, shall be equal one to the other.

a Take $AE=AD$; and *b* join CD , and BE .

Because, in the triangles ACD, ABE , are $AB = AC$, and $AE = AD$, and the angle A common to them both, *d constr.* therefore is the angle $ABE = ACD$, and the angle $AEB = ADC$, and the base $BE = CD$; also $EC = DB$. Therefore in the triangles BEC, BDC *f 3. ax.* it will be the angle $ECB = DBC$, *g 4. i.* Which was to be dem. Also therefore the angle $EBC = DCB$. but the angle $ABE = ACD$; therefore *h before.* the angle $ABC = ACB$. Which was to be dem. *k 3. ax.*

Coroll.



a 3. i.
b 1. post.

c hyp.
d constr.

e 4. i.

f 3. ax.
g 4. i.

h before.

k 3. ax.

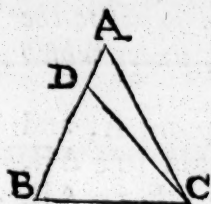
The first Book of

Coroll.

Hence, every equilateral triangle is also equiangular.

P R O P. VI.

If two angles ABC , ACB of a triangle ABC , be equal the one to the other, the sides AC , AB , subtended under the equal angles, shall also be equal one to the other.



a 3. 1.
b 1. post.
c suppos.
d hyp.
e 4. 1.
f 9. ax.

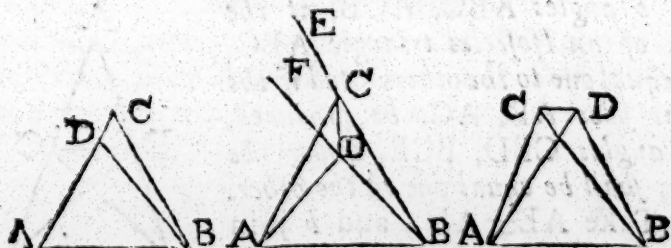
If the sides be not equal, let one be bigger than the other, suppose $BA > CA$. a Make $BD = CA$, and b draw the line CD .

In the triangles DBC , ACB , because $BD = CA$, and the side BC is common, and the angle $DBC = ACB$, the triangles DBC , ACB e shall be equal the one to the other, a part to the whole. f Which is impossible.

Coroll.

Hence, every equiangular triangle is also equilateral.

P R O P. VII.



Upon the same right line AB two right lines being drawn AC , BC , two other right lines equal to the former, AD , BD , each to each (viz.) $AD = AC$, and $BD = BC$) cannot be drawn from the same points A , B , on the same side C , to several points, as C and D , but only to C .

1. Case. If the point D be set in the line AC , it is plain that AD is not equal to AC .

2. Case. If the point D be placed within the triangle ACB , then draw the line CD , and produce BDF , and BCE . Now you would have $AD = AC$. then

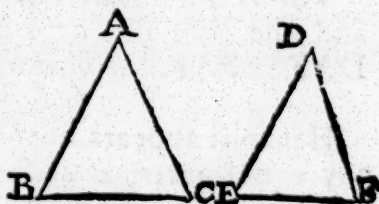
then the angle $ADC = ACD$; as also, because $BD = BC$, the angle $FDC = b ACD$. therefore is c *suppos.* the angle $FDC = d ACD$. that is, the angle $FDC = ADC$. *d Which is impossible.*

3. Case. If D falls without the triangle ACB, let CD be joined.

Again, the angle $ACD = ADC$, and the angle $BCD = BDC$. Therefore the angle $ACD = BDC$, viz. the angle $ADC = BDC$. Which is impossible. Therefore, &c.

P R O P. VIII.

If two triangles ABC, DEF have two sides AB, AC, equal to two sides DE, DF, each to each, and the base BC equal to the base EF, then the angles contained under the equal right lines shall be equal, viz. A to D.



Because $BC = EF$, if the base BC be laid on the base EF, they will agree: therefore whereas $AB = DE$, and $AC = DF$, the point A will fall on D (for it cannot fall on any other point, by the precedent proposition) and so the sides of the angles A and D are coincident; wherefore those angles are equal. Which was to be demonstrated.

Coroll.

1. Hence, triangles mutually equilateral are also mutually $\propto$ equiangular.

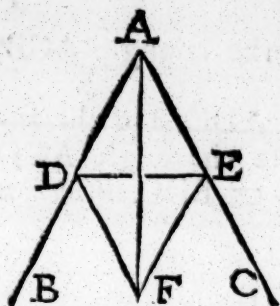
2. Triangles mutually equilateral $\propto$ are equal one to the other.

P R O P.

P R O P. IX.

a 3. 1.

b 1. 1.

c constr.
d 8. 1.

To bisect, or divide into two equal parts, a right-lined angle given BAC.

a Take $AD = AE$, and draw the line DE ; upon which b make an equilateral triangle DFE . draw the right line AF ; it shall bisect the angle.

For $AD = AE$, and the side AF is common, and the base $DF = FE$. d therefore the angle $DAF = EAF$. Which was to be done.

Coroll.

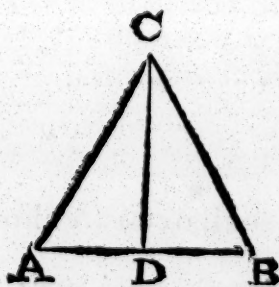
Hence it appears how an angle may be cut into any equal parts, as 4, 8, 16, &c. to wit, by bisecting each part again.

The method of cutting angles into any equal parts required, by a Rule and Compass, is as yet unknown to Geometricians.

P R O P. X.

a 1. 1.

b 9. 1.

c constr.
d 4. 1.

To bisect a right line given AB.

Upon the line given AB a erect an equilateral triangle ABC ; and b bisect the angle C with the right line CD . That line shall also bisect the line given AB.

For $AC = BC$, and the side CD is common, and the angle $ACD = BCD$. therefore $AD = BD$. Which was to be done.

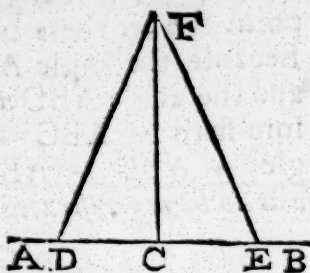
The practice of this and the precedent proposition is easily shewn by the construction of the 1. proposition of this book.

P R O P.

P R O P. XI.

From a point *C* in a right line given *AB* to erect a right line *CF* at right angles.

a Take on either side of the point given $CD=CE$, upon the right line *DE* *b* erect an equilateral triangle. draw the line *FC*, and it will be the perpendicular required.



a 3. 1.

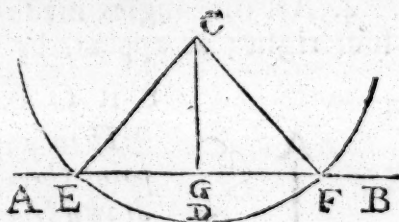
b 1. 1.

For the triangles *DFC*, *EFC* are mutually *c* equilateral; *d* therefore the angle $DCF=ECF$. *e* therefore *FC* is perpendicular. Which was to be done. *c* constr. *d* 8. 1. *e* 10. def.

The practice of this and the following is easily performed by the help of a square.

P R O P. XII.

Upon an infinite right line given *AB*, from a point given that is not in it, to let fall a perpendicular right line *CG*.



From the center *C* *a* describe a circle cutting the right line given *AB* in the points *E* and *F*. Then *b* bisect *EF* in *G*, and draw the right line *CG*, which will be the perpendicular required.

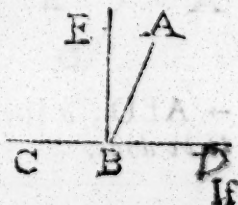
a 3. post.

b 10. 1.

Let the lines *CE*, *CF* be drawn. The triangles *EGC*, *FGC* are mutually *c* equilateral. *d* therefore the angles *EGC*, *FGC* are equal, and by *e* consequence right. *e* Wherefore *GC* is a perpendicular. Which was to be done. *c* constr. *d* 8. 1. *e* 10. def.

P R O P. XIII.

When a right line *AB* standing upon a right line *CD* maketh angles *ABC*, *ABD*; it maketh either two right angles, or two angles equal to two right.



a def. 10.

b 11. 1.

c 19. ax.

d 3. ax.

e 2. ax.

If the angles ABC , ABD be equal, a then they make two right angles; if unequal, then from the point B b let there be erected a perpendicular BE . Because the angle ABC $c =$ to a right $+ ABE$, and the angle ABD $d =$ to a right $- ABE$, therefore shall be $ABC + ABD$ $e =$ to two right angles $+ ABE - ABE = 2$ right angles. Which was to be demonstrated.

Corollaries.

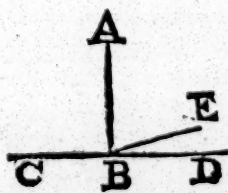
1. Hence, if one angle ABD be right, the other ABC is also right; if one acute, the other is obtuse, and so on the contrary.

2. If more right lines than one stand upon the same right line at the same point, the angles shall be equal to two right.

3. Two right lines cutting each other make angles equal to four right.

4. All the angles made about one point make four right; as appears by Coroll. 2.

P R O P. XIV.



If to any right line AB , and a point therein B , two right lines, not drawn from the same side, do make the angles ABC, ABD , on each side equal to two right, the lines CB, BD , shall make one strait line.

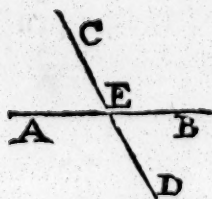
a 13. 1.

b hyp.

c 9. ax.

If you deny it, let CB, BE make one right line; then shall be the angle $ABC + ABE$ $a =$ two right angles $b = ABC + ABD$. Which is C absurd.

P R O P. XV,



If two right lines AB, CD , cut thro' one another, then are the two angles which are opposite, viz. CEB, AED , equal one to the other.

For the angle $AEC + CEB$ $a =$ to two right angles $= AEC$

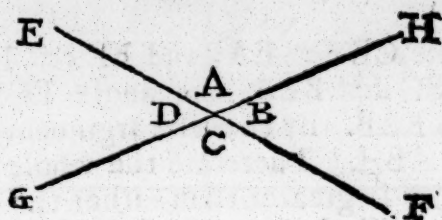
a 13. 1.

b 3. ax.

$+ AED$; b therefore $CEB = AED$. Which was to be done.

Schol.

Schol. 1.

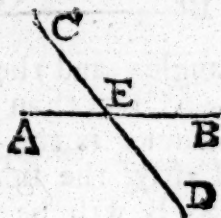


If to any right line GH, and in it a point A, two right lines being drawn EA, AF, and not taken on the same side, make the vertical (or opposite) angles D and B equal, those right lines EA, AF, do meet directly and make one strait line.

For two right angles are *a* equal to the angle *a* 13. 1. $D + A = B + A$. *b* Therefore EA, AF, are in a *b* 14. 1. strait line. Which was to be demonstrated.

Schol. 2.

If four right lines EA, EB, EC, ED, proceeding from one point E, make the angles, vertically opposite, equal the one to the other, each two lines, AE, EB, and CE, ED, are placed in one strait line.



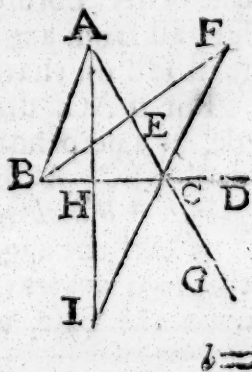
For because the angle $AEC + AED + CEB + DEB$ *a* = to four right angles, therefore the angle $AEC + AED$ *b* = $CEB + DEB$ = to two right angles. *c* Therefore CED and AEB are strait lines. Which was to be demonstrated.

a 4. c. 13. 1.
b hyp. &
c ax.
c 14. 1.

PROP. XVI.

One side BC of any triangle ABC being produc'd the outward angle ACD will be greater than either of the inward and opposite angles CAB, CBA.

Let the right lines AH, BE *a* bisect the sides AC, BC; *b* produce EF = BE, and HI,



a 10. 1. &
b 3. 1.

B

b =

$b = AH$. and join FC , and IC ; and produce ACG .

c constr.

d 15. 1.

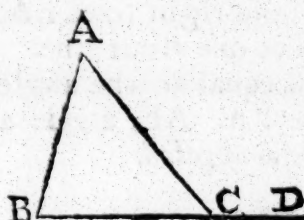
e 4. 1.

f 15. 1.

g 9. ax.

Because $CE = EA$, and $EF = EB$, and the angle $FEC = BEA$, the angle ECF shall be equal to EAB . By the like argument is the angle $ICH = ABH$. Therefore the whole angle ACD ($fBCG$) g is greater than either the angle CAB or ABC . Which was to be demonstrated.

P R O P. XVII.



Two angles of any triangle ABC , which way soever they be taken, are less than two right angles.

a 13. 1.

b 16. 1.

c 4. ax.

Let the side BC be produced. Because the angle $ACD + ACB =$ two right angles, and the angle $ACD = A$, c therefore $A + ACB >$ than two right angles. After the same manner is the angle $B + ACB >$ than two right. Lastly, the side AB being produced, the angle $A + B$ will be also less than two right angles. Which was to be demonstrated.

Coroll.



1. Hence it follows that in every triangle wherein one angle is either right or obtuse, the two others are acute angles.

2. If a right line AE make unequal angles with another right line DC , one acute AED , the other obtuse AEC , a perpendicular AD , let fall from any point A to the other line CD , shall fall on that side the acute angle is of.

* 17. 1.

For if AC , drawn on the side of the obtuse angle, be a perpendicular, then in the triangle AEC , shall $AEC + ACE$ be greater than two right angles. * Which is contrary to the precedent prop.

3. All the angles of an equilateral triangle, and the two angles of an Isosceles triangle that are upon the base, are acute.

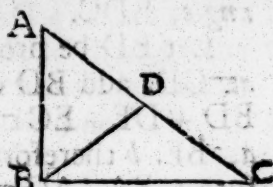
P R O P.

PROP. XVIII.

The greatest side AC of every triangle ABC subtends the greatest angle ABC.

From AC take away AD = AB, and join BD. Therefore is the angle ADB =

ABD. But ADB = C; therefore is ABD = C; therefore the whole angle ABC = C. After the same manner, shall be ABC = A. Which was to be demonstrated.



a 3. 1.
b 5. 1.

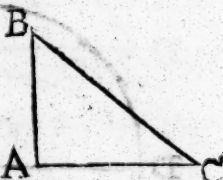
c 16. 1.
d 9. ax.

PROP. XIX.

In every triangle ABC, under the greatest angle A is subtended the greatest side BC.

For if AB be supposed equal to BC, then will be the angle A

= C, which is contrary to the Hypothesis: and if AB < BC, then shall be the angle C < A, which is against the Hypothesis. Wherefore rather BC < AB; and after the same manner BC < AC. Which was to be demonstrated.



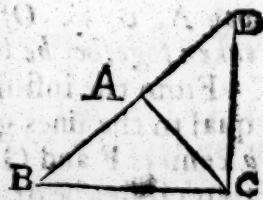
a 5. 1.
b 18. 1.

PROP. XX.

Of every triangle ABC, two sides BA, AC, any way taken, are greater than the side that remains BC.

Produce the line BA, and take AD = AC, and draw the

line DC; then shall the angle D be equal to ACD, therefore is the whole angle BCD < D; therefore BD (e BA + AC) < BC. Which was to be demonstrated.



a 3. 1.

b 5. 1.
c 9. ax.
d 19. 1.
e constr. &c
2 ax.

PROP. XXI.

If from the utmost points of one side BC, of a triangle ABC, two right lines BD, CD, be drawn to any point within the triangle, then are both those two lines shorter than the two other sides of



B 2

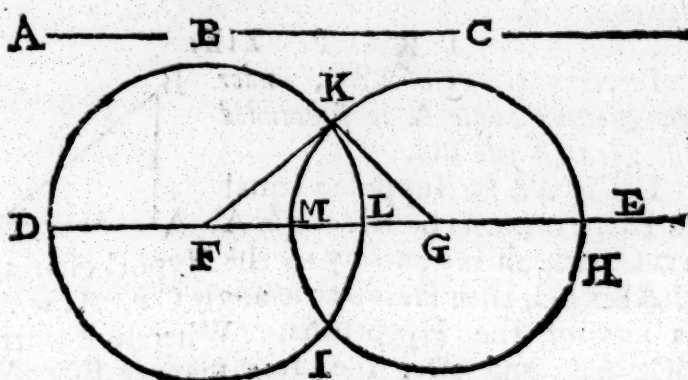
the

The first Book of

the triangle, BA, CA; but do contain a greater angle, BDC.

- a 20. 1. Let BD be produced to E. Then is $CE + ED = CD$. add BD common to both, b then shall be $BD + DE + EC = CD + BD$. Again, $BA + AE = BE$. b therefore $BA + AC = BE + EC$. Wherefore 1. $BA + AC = BD + DC$. 2. The angle BDC $\angle DEC \angle A$. Therefore the angle BDC $\angle A$. Which was to be demonstrated.

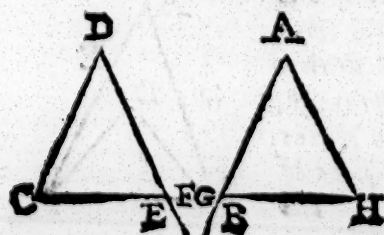
P R O P. XXII.



To make a triangle FKG of three right lines FK, FG, GK, which shall be equal to three right lines given A, B, C. Of which it is necessary that any two taken together be longer than the third.

- a 3. 1. From the infinite line DE a take DF, FG, GH, equal to the lines given A, B, C. Then if from the b centers F and G by the distances of FD and GH, two circles be drawn cutting each other in K, and the right lines KF, KG be joined, the triangle FKG shall be made, c whose sides FK, FG, GK are equal to the three lines DF, FG, GH, d that is, to the three lines given A, B, C. Which was to be done.

P R O P. XXIII.

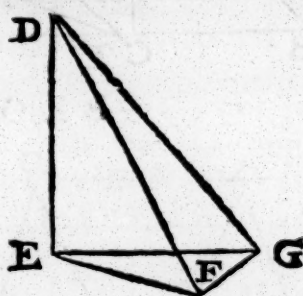
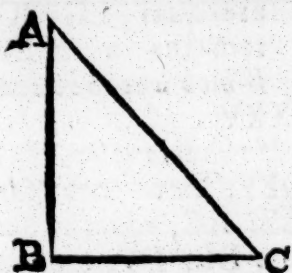


At a point A in a right line given AB, to make a right-lined angle A equal to a right-lined angle given D.

- a 1. post. a Draw the right line

Line CF cutting the sides of the angle given any ways; *b* make $AG = CD$; upon AG *c* raise a triangle equilateral to the former CDF , so that AH be equal to DF , and GH to CF . then shall you have the angle $A = D$. Which was to be done. *d* 8. 1.

P R O P. XXIV.

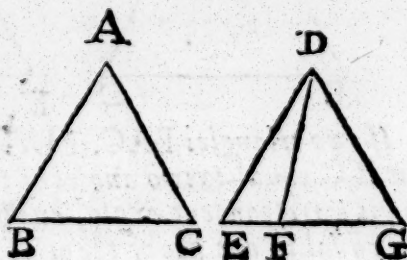


If two triangles ABC , DEF have two sides of the one triangle AB , AC equal to two sides of the other triangle DE , DF , each to other, and have the angle A greater than the angle EDF contained under the equal right lines, they shall have also the base BC greater than the base EF .

a Let the angle EDG be made equal to A , and the side $DG = DF = AC$; and let EG , and FG be joined. *c* hyp.

1. Case. If EG fall above EF ; Because $AB = DE$, and $AC = DG$, and the angle $A = EDG$, therefore is $BC = EG$. But because $DF = DG$, therefore is the angle $DFG = DGF$; is the angle $DFG > EFG$, and by consequence the angle $EFG < EGF$. wherefore $EG (BC) > EF$.

2. Case. If the base EF falls in the same place with EG , it is evident that $EG (BC) > EF$.

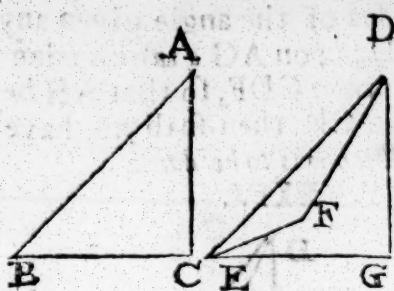


B 3

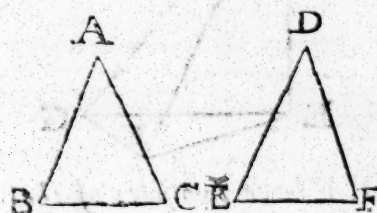
3. Case.

m 21. 1.

n 5. ix.



P R O P. XXV.



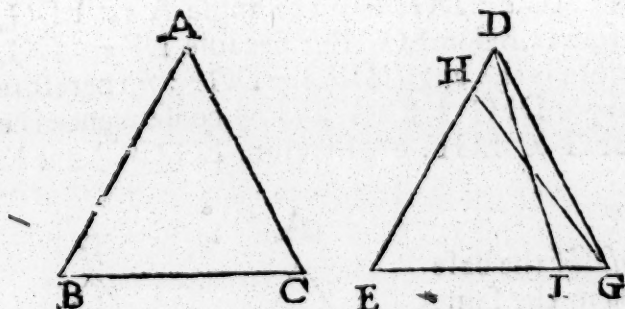
have the angle A contained under the equal right lines greater than the angle D.

a 4. 1.

b 24. 1.

For if the angle A be said to be equal to D, a then is the base $BC = EF$, which is against the Hypothesis. If it be said the angle $A > D$, then b will be $BC > EF$, which is also against the Hypothesis. Therefore $BC = EF$. Which was to be dem,

P R O P. XXVI.



If two triangles BAC, EDG have two angles of the one B, C, equal to two angles of the other E, DGE, each to his correspondent angle, and have also one side of the one equal to one side of the other, either that side which lyeth betwixt the equal angles, or that which is subtended under one of the equal angles; the other sides al-

3. Case. If EG fall below EF, then because $DG + GE = DF + FE$, if from both be taken away DG, DF which are equal; EG (BC) remains $= EF$. Which was to be dem.

If two triangles ABC, DEF, have two sides AB, AC, equal to two sides DE, DF, each to other, and have the base BC greater than the base EF, they shall also

so of the one shall be equal to the other sides of the other, each to his correspondent side, and the other angle of the one shall be equal to the other angle of the other.

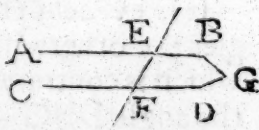
1. Hypothesis. Let BC be equal to EG, which are the sides that lie between the equal angles. Then I say $BA = ED$, and $AC = DG$, and the angle $A = EDG$. For if it be said that $ED < BA$, then a let a 3. 1. EH be made equal to BA, and let the line GH be drawn.

Because AB $b = HE$, and BC $c = EG$, and the angle B $c = E$, therefore shall be the angle EGH $d = CE = DGE$. f Which is absurd. After the same manner let AC be equal to DG, d then will the angle A be equal to EDG. e hyp. f 9. ax.

2. Hyp. Let AB be equal to ED. Then I say BC = EG, and $AC = DG$, and the angle $A = EDG$. For if EG be greater than BC, make EI = BC, and join the line DI. Now because AB $g = ED$, and BC $h = EI$, and the angle G $g = E$; therefore will be the angle EID $k = C l = EGD$. m Which is absurd. Therefore is BC = EG, and so as before $AC = DG$, and the angle $A = EDG$. Which was to be dem. g hyp. h suppos. k 4. 1. l hyp. m 16. 1.

P R O P. XXVII.

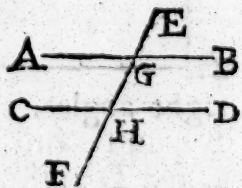
If a right line EF, falling upon two right lines AB, CD, make the alternate angles AEF, DFE, equal the one to the other, then are the right lines AB, CD, parallel.



If AB, CD be said not to be parallel, produce them till they meet in G, which being supposed, the outward angle AEF will be a greater than a 16. 1. the inward angle DFE, to which it was equal by Hypothesis. Which is repugnant.

P R O P. XXVIII.

If a right line EF, falling upon two right lines AB, CD, make the outward angle AGE of the one line equal to CHG the inward and opposite angle of the other on the same side, or make the inward



B 4

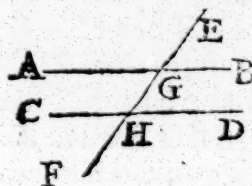
angles

angles on the same side AGH, CHG, equal to two right angles, then are the right lines AB CD, parallel.

Hyp. 1. Because by Hypothesis the angle AGE = CHG, *a* therefore are BHG, CHG alternate angles and equal: And *b* therefore are AB and CD parallel

Hyp. 2. Because by Hypothesis the angle AGH + CHG = to two right *a* = AGH + BGH, *b* thence is CHG = BGH; and *c* therefore AB, CD, are parallel. Which was to be demonstrated.

P R O P. XXIX.



If a right line EF fall upon two parallels, AB, CD, it will make both the alternate angles DHG, AGH, equal each to other, and the outward angle BGE equal to the inward and opposite angle on the same side DHG, as also the inward angles on the same side AGH, CHG equal to two right angles.

It is evident that AGH + CHG = two right angles; *a* otherwise AB, CD, would not be parallel, which is contrary to the Hypothesis: But moreover the angle DHG = CHG *b* two right; therefore is DHG *c* = AGH *d* = BGE. Which was to be dem.

Coroll.



Hence it follows that every parallelogram AC having one angle right A, the rest are also right.

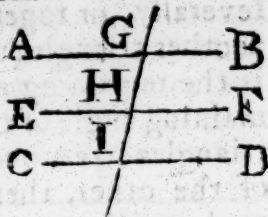
a 29 1.
b 3. ax.

For A = B *a* = two right angles. Therefore, whereas A is right, *b* B must be also right. By the same argument are C and D right angles.

P R O P.

PROP. XXX.

Right lines (AB, CD) parallel to one and the same right line EF, are also parallel the one to the other.

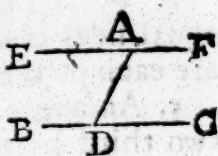


Let GI cut the three right lines given any ways. then because AB, EF, are parallel, the angle AGI will be $a = \text{EHI}$. also because CD and EF are parallel, the angle EHI will be $a = \text{DIG}$. *b* Therefore the angle AGI = DIG. *c* whence AB and CD are parallel. Which was to be demonstrated.

a 29. 1.
b 1. ax.
c 27. 1.

PROP. XXXI.

From a point given A to draw a right line AE, parallel to a right line given BC.

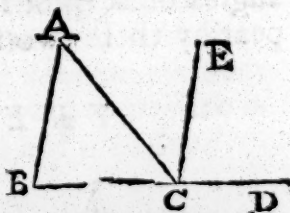


From the point A draw a right line AD to any point of the given right line; with which at the point thereof a A make an angle DAE = ADC. *b* then will AE and BC be parallel. Which was to be done.

a 23. 1.
b 27. 1.

PROP. XXXII.

Of any triangle ABC one side BC being drawn out, the outward angle ACD shall be equal to the two inward opposite angles A, B, and the three inward angles of the triangle, A, B, ACB, shall be equal to two right angles.



From C a draw CE parallel to BA. Then is the angle A $b = \text{ACE}$, and the angle B $b = \text{ECD}$. *b* Therefore A + B $c = \text{ACE} + \text{ECD} d = \text{ACD}$. Which was to be demonstrated.

d 19. ax.

I affirm $\text{ACD} + \text{ACB} e = \text{two right angles}$; *e* therefore A + B + ACB = two right angles. Which *f* was to be demonstrated.

e 13. 1.

f 1. ax.

Coroll.

1. The three angles of any triangle taken together are equal to three angles of any other triangle taken together. From whence it follows,

2. That

2. That if in one triangle, two angles (taken severally, or together) be equal to two angles of another triangle (taken severally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the sum of the remaining angles of the one triangle equal to the sum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right. Likewise, that angle in a triangle which is equal to the other two, is it self a right angle.

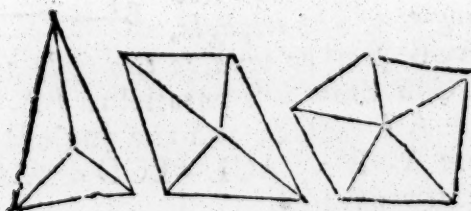
4. When in an Iſosceles the angle made by the equal ſides is right, the other two upon the baſe are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle. For $\frac{2}{3}$ of two right angles is equal to $\frac{4}{3}$ of one.

Schol.

By the help of this propoſition you may know how many right angles the inward and outward angles of a right-lined figure make ; as may appear by theſe two following Theoremes.

THEOREME I.



All the angles of a right-lined figure do together make twice as many right angles, abating four, as there are ſides of the figure.

From any point within the figure let right lines be drawn to all the angles of the figure, which shall

shall resolve the figure into as many triangles as there are sides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are sides. But the angles about the said point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the said point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, abating four, as there are sides of the figure. *Which was to be demonstrated.*

Coroll.

Hence, all right-lined figures of the same species have the sums of their angles equal.

THEOREM II.

All the outward angles of any right-lined figure, taken together, make up four right angles.

For all the several inward angles of a figure, with the several outward angles of the same, make two right angles; therefore all the inward angles together with all the outward, make twice as many right angles as there are sides of the figure; but (as it was now shewn) all the inward angles, with four right, make twice as many right as there are sides of the figure; therefore the outward angles are equal to four right angles. *Which was to be dem.*

Coroll.

All right-lined figures, of whatsoever species, have the sums of their outward angles equal.

PROP. XXXIII.

If two equal and parallel lines AB, CD, be joined together with two other right lines AC, BD, then are those lines also equal and parallel.



Draw a line from C to B. Now because AB and CD are parallel, and the angle $\angle ABC = \angle BCD$; a 29. 1. and also by hypothesis $AB = CD$, and the side CB common, therefore is $AC = BD$, and the angle $\angle ACB$

b 4. r.
c 27. r.

ACB $b = DBC$. c whence also AC, BD, are parallel.
P R O P. XXXIV.



In parallelograms, as ABCD, the opposite sides AB, CD, and AC, BD, are equal each to other; and the opposite angles A, D, and ABD, ACD, are also equal; and the diameter BC bisects the same.

a hyp.

b 29. r.
c 2. ax.

d 26. r.

Because AB, CD a are parallel, b therefore is the angle $ABC = BCD$. Also because AC, BD, are a parallel, b therefore is the angle $ACB = CBD$; c therefore the whole angle $ACD = ABD$. After the same manner is $A = D$. Moreover because the angles ABC, ACB lie at each end of the side CB, and are equal to BCD, CBD, d therefore is $AC = BD$ and $AB = CD$, and so the triangle $ABC = CBD$. Which was to be demonstrated.

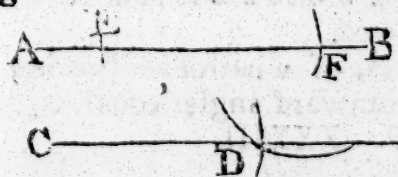
Schol.

Every four-sided figure ABDC having the opposite sides equal, is a parallelogram.

a 17. r.

For by 8. r. the angle $ABC = BCD$; a wherefore AB, CD, are parallel. In like manner is the angle $BCA = CBD$; a wherefore AC, BD are also parallel. b Therefore ABCD is a parallelogram. Which was to be demonstrated.

b 35. def. 1.



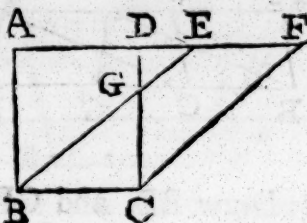
From hence may be learned how to draw a parallel CD to a right line given AB, thro' a point assigned C.

Take in the line AB any point, as E. From the centers E and C at any distance draw two equal circles EF, CD. From the center F by the space of EC draw a circle FD, which shall cut the former circle CD in the point D. Then shall the line drawn CD be parallel to AB. for, as it was before demonstrated, CEFD is a parallelogram.

P R O P.

P R O P. XXXV.

Parallelograms, BCDA, BCFE, which stand upon the same base BC, and between the same parallels AF, BC, are equal one to the other.

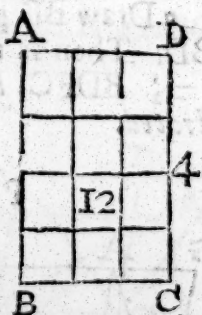


For AD $a = BC$ $a = EF$. a 34. 1.
add DE common to both, b then is $AE = DF$. b 2. ax.
But also $AB = DC$, and the angle $A = CDF$. c 29. 1.
Therefore is the triangle $ABE = DCF$. take away d 4. 1.
 DGE common to both triangles, e then is the Trapezium $ABGD = EGCF$. e 3. ax.
add BGC common to both, f then is the parallelogram $ABCD = EBCF$. f 2. ax.
Which was to be demonstrated.

The demonstration of any other cases is not unlike, but much more plain and easy.

Schol.

If the side AB of a right-angled parallelogram ABCD be conceived to be carried along perpendicularly thro' the whole line EC, or BC thro' the whole line AB, the area or content of the rectangle ABCD shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing or multiplication of two contiguous sides. For example; let AB be supposed four foot, and BC three: draw three into four, there will be produced twelve square feet for the area of the rectangle.



This being supposed, the dimension of any parallelogram (*ABCD) is found out by this theorem. For the area thereof is produced from the altitude BA drawn into the base BC. So the area of the parallelogram $AC = ABCD$, is made by the drawing of BA into BC, therefore, &c.

* See the fig. of prop. 35.

P R O P.

P R O P. XXXVI.



Parallelograms BCDA, GHFE, standing upon equal bases BC, GH, and betwixt the same parallels AF, BH, are equal one to the other.

a hyp.

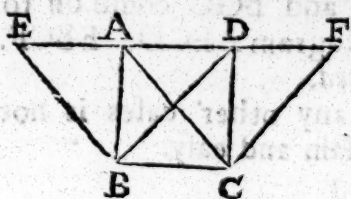
b 34. 1.

c 33. 1.

d 35. 1.

Draw BE, and CF. Because BC $a = GH$ $b = EF$, c therefore is BCFE a parallelogram. Whence the parallelogram BCDA $d = BCFE$ $d = GHFE$. Which was to be demonstrated.

P R O P. XXXVII.



Triangles, BCA, BCD, standing upon the same base BC, and between the same parallels BC, EF, are equal the one to the other.

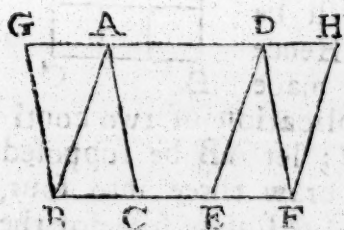
a 31. 1.

b 34. 1.

c 35. 1.

a Draw BE parallel to CA, a and CF parallel to BD. Then is the triangle BCA $b = \frac{1}{2}$ Pgr. BCAF $c = \frac{1}{2}$ BDFC $b = BCD$. Which was to be demonstrated and 7. ax. strated.

P R O P. XXXVIII.



Triangles, BCA, EFD, set upon equal bases BC, EF, and between the same parallels GH, BF, are equal the one to the other.

a 34. 1.

b 36. 1.

c 34. 1.

c 34. 1.

Draw BG parallel to CA, and FH parallel to ED. Then is the triangle BCA $a = \frac{1}{2}$ Pgr. BCAG $b = \frac{1}{2}$ EDHF $c = EFD$. Which was to be demonstrated and 7. ax. strated.

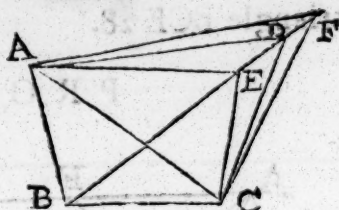
Schol.

If the base BC be greater than EF, then is the triangle BAC $\square$ EDF, and so on the contrary.

P R O P.

P R O P. XXXIX.

Equal triangles BCA , BCD , standing on the same base BC , and on the same side, they are also between the same parallels AD , BC .



If you deny it, let another line AF be parallel to BC ; and let CF be drawn. Then is the triangle CBF $a = CBA$ $b = CBD$. *c Which is absurd.*

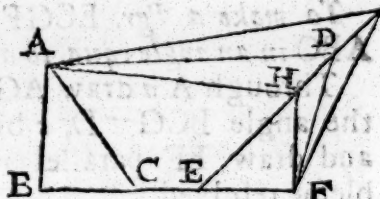
a 37. 1.

b hyp.

c 9. ax.

P R O P. XL.

Equal triangles BCA , EFD , standing upon equal bases BC , EF , and on the same side, they are betwixt the same parallels.



If you deny it, let another line AH be parallel to BF , and let FH be drawn. Then is the triangle EFH $a = BCA$ $b = EFD$. *c Which is absurd.*

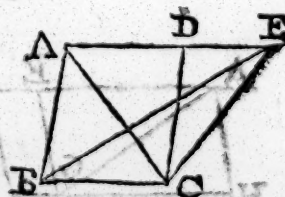
a 38. 1.

b hyp.

c 9. ax.

P R O P. XLI.

If a Pgr. $ABCD$ have the same base BC with the triangle BCE , and be between the same parallels AE , BC , then is the Pgr. $ABCD$ double to the triangle BCE .



Let the line AC be drawn. Then is the triangle BCA $a = BCE$. therefore is the Pgr. $ABCD$ $b = 2 BCA$ $c = 2 BCE$. *Which was to be demonstrated.*

a 37. 1.

b 34. 1.

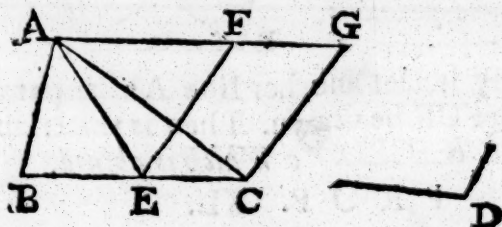
c 6. ax.

Schol.

From hence may the area of any triangle BCE be found, for whereas the area of the Pgr. $ABCD$ is produced by the altitude drawn into the base, therefore shall the area of a triangle be produced by half the altitude drawn into the base, or half the

the base drawn into the altitude, as the base BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

P R O P. XLII.

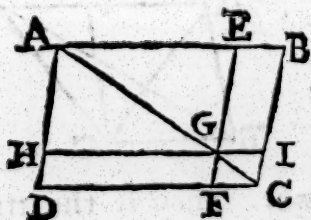


To make a Pgr. $ECGF$ equal to a triangle given ABC in an angle equal to a right-lined angle given D .

- a 31. 1. Through A draw AG parallel to BC .
 b 23. 1. make the angle $BCG = D$,
 c 10. 1. bisect the base EC in E , and draw EF parallel to CG . then is the probleme resolved.

- d 38. 1. For AE being drawn, the angle ECG is equal to D by construction, and the triangle EAC
 e 41. 1. $AEC =$ Pgr. $ECGF$. Which was to be done.

P R O P. XLIII.

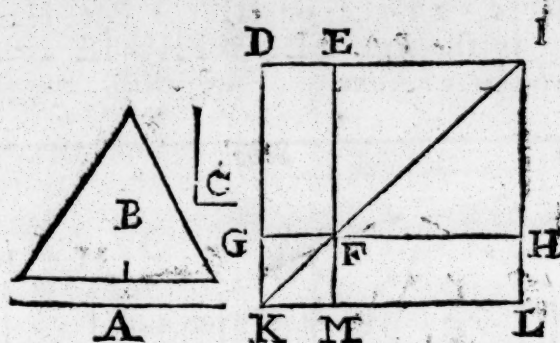


In every Pgr. $ABCD$, the complements DG , GB , of those Pgrs. HE , FI , which stand about the diameter, are equal one to the other.

- a 34. 1. For the triangle ACD
 b 3. ax. $a = ACB$, and the triangle AGH
 $a = AGE$, and the triangle GCF
 $a = GCI$ Therefore the Pgr. $DG = BG$. Which was to be demonstrated.

P R O P.

PROP. XLIV.

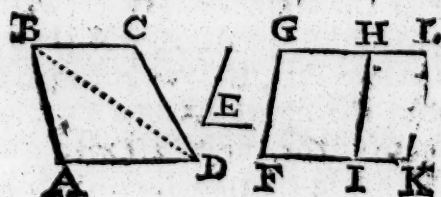


Upon a right line given A, to make a Parallelogram FL at a right-lined angle given C, equal to a triangle given B.

a Make a Pgr. $FD =$ to the triangle B. so that the angle GFE may be equal to C. Produce GF till FH be equal to the line given A. through H draw IL parallel to EF, which let DE produced meet in the point I. let DG drawn forth meet with a right line drawn from I in the point K. thro' K draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then shall FL be the Pgr. required.

For the Pgr. $FL = FD = B$, and the angle $MFH = GFE = C$. Which was to be done.

PROP. XLV.



Upon a right line given FG to make a Pgr. FL equal to a right-lined figure given ABCD, at a right-lined angle given E.

a 44. I.

Resolve the right-lined figure given into triangles BAD, BCD. then *a* make a Pgr. $FH = BAD$, so that the angle F may be equal to E FI being produced, *a* make on HI the Pgr. $IL = BCD$.

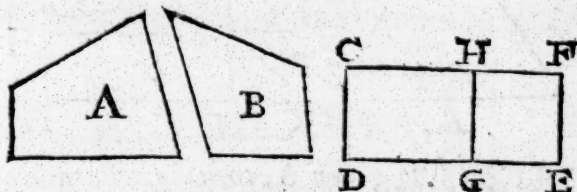
b 19. ax.

Then is the Pgr. $FL = FH + IL = ABCD$.

c constr.

Which was to be done.

Schol.

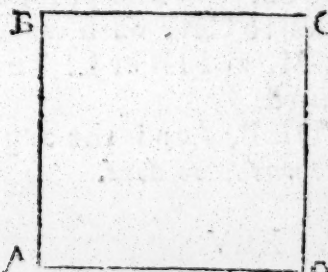


Hence is easily found the excess, HE, whereby any right-lined figure, A, exceeds a less right-lined figure, B; namely, if to some right line, CD, both be applied, Pgr. $DF = A$, and $DH = B$.

P R O P. XLVI.

a 11. I.

b 3. I.



Upon a right line given AD to describe a square AC.

a Erect two perpendiculars AB, DC, *b* equal, to the line given AD; then join BC, and the thing required is done.

c constr.

d 28. I.

e constr.

f 34. I.

g sch. 29. I.

h 29. def.

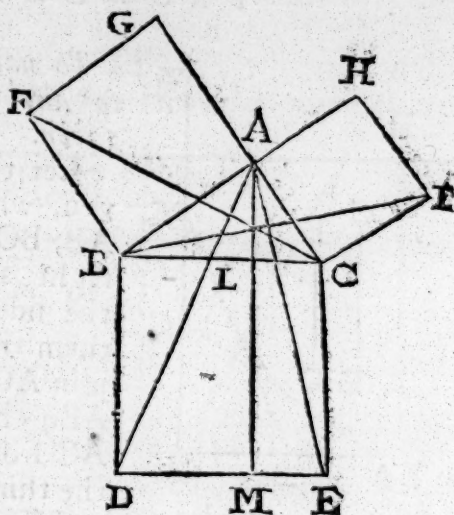
For, whereas the angle $A + D =$ two right, *d* therefore are AB, DC parallel. But they are also *e* equal; *f* therefore AD, BC are both parallel and equal; therefore the figure AC is a Pgr. and equilateral. Moreover the angles are all right, *g* because one, A, is right; *h* therefore AC is a square. Which was to be done.

After the same manner you may easily describe a rectangle contained under two right lines given.

P R O P.

P R O P. XLVII.

In right-angled triangles, BAC, the square BE, which is made of the side BC that subtends the right angle BAC, is equal to both the squares BG, CH, which are made of the sides AB, AC containing the right angle.



Join AE, and AD; and draw AM parallel to CE.

Because the angle DBC $a = FBA$, add the angle ABC common to them both; then is the angle ABD $= FBC$. Moreover AB $b = FB$, and BD $b = BC$; therefore is the triangle ABD $= FBC$. But the Pgr. BM $d = 2$ ABD, and the Pgr. $dBG = 2$ FBC (for GAC is one right line by Hypothesis, and 14. 1.) therefore is the Pgr. BM $= BG$. By the same way of argument is the Pgr. CM $= CH$. Therefore is the whole BE $= BG + CH$. Which was to be demonstrated.

Schol.

This most excellent and useful theoreme hath deserved the title of *Pythagoras* his theoreme, because he was the inventor of it. By the help of which the addition and subtraction of squares are performed; to which purpose serve the two following problemes.

C 2

P R O

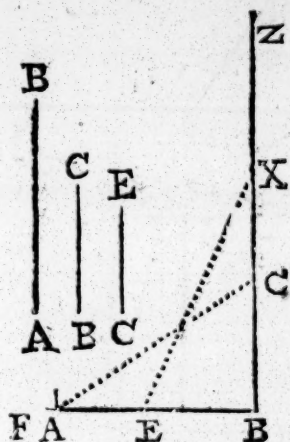
P R O B L E M I.

Andr.
Tacq.

2. II. I.

b 47. r.

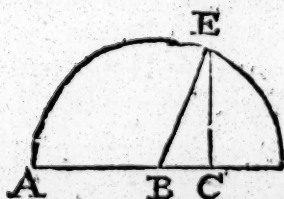
C 2. 8x.



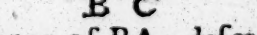
To make one square equal
to any number of squares
given.

Let three squares be given, whereof the sides are AB, BC, CE. *a* Make the right angle FBZ having the sides infinite; and on them transfer BA and BC; join AC. then is $ACq = ABq + BCq$. Then transfer AC from B to X, and CE the third side given from B to E; join EX. *b* Then is $EXq = EBq (CBq) + BXq (ACq)$ *c* $= CEq + ABq + BCq$. Which was to be done.

PROBLEM II.



Two unequal right lines being given AB, BC, to make a square equal to the difference of the two squares of the given lines, AB, BC.


 From the center B, at the distance of BA, describe a circle; and from the point C erect a perpendicular CE meeting with the circumference in E; and draw BE. *a* Then is $BE^2 (BA^2) = BC^2 + CE^2$. *b* Therefore $BA^2 - BC^2 = CE^2$. Which was to be done.

2 47. 1.

b. 3. ax.

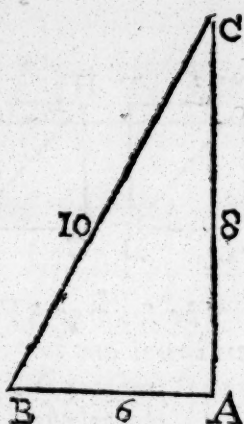
P R O-

PROBLEM III.

Any two sides of a right-angled triangle ABC being known, to find out the third.

Let the sides AB, AC, encompassing the right angle, be, the one 6 foot, the other 8. Therefore, whereas $AC^2 + AB^2 = 64 + 36 = 100 = BC^2$, thence is $BC = \sqrt{100} = 10$.

Otherwise, let the sides AB, BC be known, the one 6 foot, the other 10. Therefore since $BC^2 - AB^2 = 100 - 36 = 64 = AC^2$, thence is $AC = \sqrt{64} = 8$. Which was to be done.

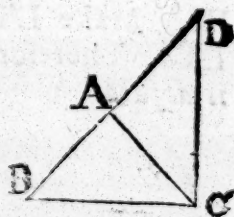


47. 1.

47. 1.

PROP. XLVIII.

If the square made upon one side BC of a triangle be equal to the squares made on the other sides of the triangle, AB, AC, then the angle BAC comprehended under two other sides of the triangle AB, AC, is a right angle.



Draw to the point A in AC a perpendicular line $DA = AB$, and join CD.

Now is $CD^2 = AD^2 + AC^2 = AB^2 + AC^2 = BC^2$. * Therefore is $CD = BC$. And therefore * See the following the triangles CAB, CAD, are mutually equilateral. Wherefore the angle $CAB = CAD =$ a right angle. Which was to be demonstrated. b 8. 1. c constr.

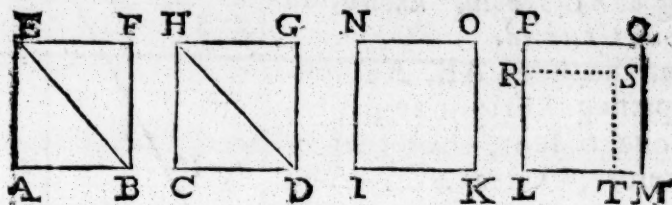
Schol.

We assumed in the demonstration of the last proposition, $CD = BC$, because CD^2 was equal to BC^2 : Our assumption we prove by the following theorem.

C 3

THEO.

THEOREM.



The squares AF, CG, of equal right lines AB, CD, are equal one to the other: And the sides IK, LM, of equal squares NK, PM, are equal one to the other.

1. Hypothesis. Draw the diameters EB, HD. Then it is evident that AF is a equal to the triangle EAB twice taken, and b equal to the triangle HCD twice taken, and equal to a CG. Which was to be demonstrated.

2. Hyp. If it may be, let LM be greater than IK. Make $LT=IK$, and let LS be a equal to LT . Therefore is $LS=b=NK=c=LQ$. Which is absurd.

d 9. ax.

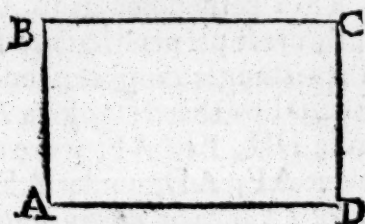
Coroll.

After the same manner any rectangles equilateral one to another, are demonstrated to be also equal,

The End of the first Book.

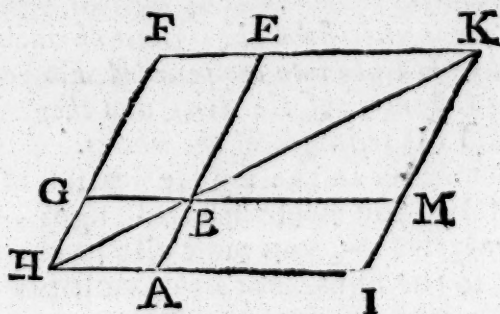
The SECOND BOOK
OF
EUCLIDE'S
ELEMENTS.

Definitions.



I. **E** Very right angled Parallelogram ABCD is said to be contained under two right lines AB, AD, comprehending a right angle.

Therefore when you meet with such as these, the rectangle under BA, AD, or more briefly the rectangle BAD, or $BA \times AD$ (or ZA, for $Z \times A$) the rectangle meant is that which is contained under the right lines BA, AD, set at right angles.



II. In every Pgr. FHIK, any one of those parallelograms which are about the diameter together with

The second Book of

with its two complements is called a Gnomon. As the Pgr. $FB + BI + GA$ (EHM) is a Gnomon; and likewise the Pgr. $FB + BI + EM$ (GKA) is a Gnomon.

P R O P. I.



If two right lines AF, AB, be given, and one of them AB divided into as many parts or segments as you please; the rectangle comprehended under the two whole right lines AB, AF, shall be equal to all the rectangles contained under the whole line AF, and the several segments, AD, DE, EB.

a 11. 1.

b 9. ax. 1.

c 34. 1.

a Set AF perpendicular to AB. Thro' F a draw an infinite line FG perpendicular to AF. From the points D, E, B, erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is *b* equal to the rectangles AH, DI, EG, that is (because DH, EI, AF, *c* are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. Which was to be demonstrated.

Schol.

If two right lines given be both divided into how many parts soever, the product of the whole multiplied into it self shall be the same with that of the parts multiplied into themselves.

a 1. 2.

b 2. ax.

For let Z be $= A + B + C$, and Y $= D + E$; then, because DZ $a = DA + DB + DC$, and EZ $a = EA + EB + EC$; and YZ $a = DZ + EZ$, *b* shall ZY be $= DA + DB + DC + EA + EB + EC$. Which was to be demonstrated.

From hence is understood the manner of multiplying compounded right lines into compounded. For you must take all the rectangles of the parts, and they will present you with the rectangle of the wholes.

But whensoever in the multiplication of lines into themselves you meet with these signs — intermingled with these +, you must also have particular regard to the signs. For of + multiply'd into — ariseth —; but of — into — ariseth +. ex. gr. let + A be to be multiply'd into B — C; then because + A is

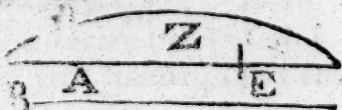
A is not affirmed of all B, but only of a part of it, whereby it exceeds C, therefore AC must remain denied; so that the product will be $AB - AC$. Or thus; because B consists of the parts C and $B - C$, * thence $AB = AC + A \times B - C$, take away AC * 1. 3. from either, then $AB - AC = A \times B - C$. In like manner, if -A be to be multiply'd into $B - C$ then, seeing by reason of the figure-, that A is not denied of all B, but only of so much as it exceeds C, therefore AC must remain affirmed, whence the product will be $-AB + AC$. Or thus; because $AB = AC + A \times B - C$; take away all throughout, and there will be $-AB = AC - A \times B + C$; add AC to either, and there will be $-AB + AC = A \times B - C$.

This being sufficiently understood, the nine following propositions, and innumerable others of that kind, arising from the comparing of lines multiply'd into themselves (which you may find done to your hand in *Vieta* and other analytical Writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, * it appears that the product of * 19. ax. any magnitude, multiply'd into the parts of any number is equal to the product of the same multiply'd into the whole number; As $5A + 7A = 12A$, and $4A \times 5A + 4A \times 7A = 4A \times 12A$. Wherefore what is here deliver'd of the multiplying of right lines into themselves, the same may be understood of the multiplying of numbers into themselves, so that whatsoever is affirmed concerning lines in the nine following theorems, holds good also in numbers; seeing they all immediately depend and are deriv'd from this first.

PRO P. II.

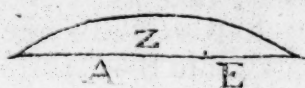
If a right line Z be divided any wise into two parts, the rectangles comprehended under the whole line Z, and each of the segments A, E, are equal to the square made of the whole line Z.



The second Book of

I say that $ZA + ZE = Zq$. For take $B = Z$;
 a 1. 2. then is $BA + BE = BZ$, that is (because $B = Z$)
 $ZA + ZE = Zq$. Which was to be demonstrated.

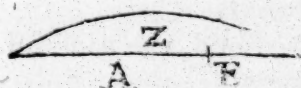
P R O P. III.



If a right line Z be divided any wise into two parts, the rectangle comprehended under the whole line Z , and one of the segments E , is equal to the rectangle made of the segments A , E , and the square described on the said segment E .

a 1. 2. I say $ZE = AE + Eq$. a For $EZ = EA + Eq$.

P R O P. IV.



If a right line Z be cut any wise into two parts, the square made of the whole line Z , is equal both to the squares made of the segments A , E , and to twice a rectangle made of the parts A , E .

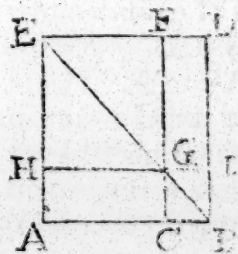
a 3. 2.

I say that $Zq = Aq + Eq + 2AE$. For $ZA = Aq + AE$, and $ZE = Eq + EA$. Therefore whereas $ZA + ZE = Zq$, thence is $Zq = Aq + Eq + 2AE$, Which was to be demonstrated.

b 2. 2.

c 1. ax.

P R O P. V.



Otherwise thus; Upon the right line AB make the square AD , and draw the diameter EB ; thro' C , the point wherein the line AB is divided, draw the perpendicular CF ; and thro' the point G draw HI parallel to AB .

d 4. cor.

32. 1.

e 32. 1.

f 6. 1.

g 34. 1.

h 29. def.

k 19. ax. 1.

Because the angle $EHG = A$ is a right angle, and AEB is half a right, therefore is the remaining angle HGE half a right angle. Therefore is $HE = HG = EF = AC$, so that HF is the square of the right line AC . After the same manner is CI proved to be CBq . Therefore AG , GD , are rectangles under AC , CB , wherefore the whole square $AD = ACq + CBq + 2ACB$. Which was to be demonstrated.

Coroll.

Coroll.

1. Hence it appears that the Pgrs which are about the diameter of a square are also squares themselves.

2. That the diameter of any square bisects its angles.

3. That if $A = \frac{1}{2} Z$, then is $Zq = 4 Aq$, and $Aq = \frac{1}{4} Zq$. As on the contrary, if $Zq = 4 Aq$, then is $A = \frac{1}{2} Z$.

P R O P. V.

If a right line AB be $A \text{---} \overset{1}{C} \text{---} \overset{1}{D} \text{---} B$
cut into equal parts AC, CB, and into unequal parts AD, DB, the rectangle comprehended under the unequal parts AD, DB, together with the square that is made of the difference of the parts CD, is equal to the square that is made of the half line CB.

I say that $CBq = ADB + CDq$.

For these are $\left\{ \begin{array}{l} CBq. \\ a CDq + CDB + DBq + CDB. \\ \text{all equal; } \left\{ \begin{array}{l} CDq + b CBD (c AC \times BD) + CDB. \\ CDq + d ADB. \end{array} \right. \end{array} \right.$

a 4. 1.
b 3. 2.
c hyp.
d 1. 2.

This theorem is somewhat differently express'd and more easily demonstrated thus; A Rectangle made of the sum and the difference of two right lines A, E, is equal to the difference of the squares of those lines.

For if $A + E$ be multiply'd into $A - E$, there ariseth $Aq - AE + EA - Eq = Aq - Eq$. Which was to be demonstrated.

Schol.

If the line AB be divided otherwise, (*viz.*) $A \text{---} C \text{---} E \text{---} D \text{---} B$
nearer to the point of bisection, in E; Then is $AEB \sqsubset ADB$.

For $AEB = CBq - CEq$, and $ADB = CBq - CDq$. Therefore, whereas $CDq \sqsubset CEq$, thence is $AEB \sqsubset ADB$. Which was to be demonstrated.

Coroll.

Coroll.

b 4. 2. 1. Hence is $ADq + DBq \sqsubset AEq + EBq$. For $ADq + DBq + 2 ADBb = ABqb = AEq + EBq + 2 AEB$. Therefore because $2 AEB \sqsubset 2 ADB$, thence is $ADq + DBq \sqsubset AEq + EBq$. Which was to be demonstrated.

c 3. ax. 2. Hence is $ADq \mid DBq - AEq \mid EBq = 2 AEB - 2 ADB$.

P R O P. VI.



If a right line A be divided into two equal parts, and another right line E,

added to the same directly in one right line, then the rectangle comprehended under the whole and the line added, (viz. $A + E$.) and the line added E, together with the square which is made of $\frac{1}{2}$ the line A, is equal to the square of $\frac{1}{2} A + E$ taken as one line.

a 4. & 3. I say that $\frac{1}{2} Aq (a Q \frac{1}{2} A) \mid AE + Eq = Q. \frac{1}{2}$

Cor. 4. 2. $A + E. a$ For, $Q \frac{1}{2} A + E = \frac{1}{2} Aq + Eq + AE$. Which was to be demonstrated.

Coroll.

Hence it follows, that if 3 right lines E, $E + \frac{1}{2} A$, $E + A$ be in arithmetical proportion, then the rectangle contained under the extreme terms E, $E + A$, together with the square of the difference $\frac{1}{2} A$, is equal to the square of the middle term $E + \frac{1}{2} A$.

P R O P. VII.



If a right line Z be divided any wise into two parts, the square of the whole line Z,

together with the square made of one of the segments E, is equal to a double rectangle comprehended under the whole line Z, and the said segment E, together with the square made of the other segment A.

a 4. 2.

b 3. 2.

I say that $Zq + Eq = 2 ZE + Aq$. For $Zq = Aq + Eq + 2 AE$, and $2 ZE = 2 Eq + 2 AE$. Which was to be demonstrated.

Coroll.

Coroll.

Hence it follows that the square of the difference of any two lines Z, E, is equal to the squares of both the lines less by a double rectangle comprehended under the said lines.

c For $Zq + Eq - 2 ZE = Aq = Q. Z - E.$

c 7. 2. and
3. ax.

P R O P. VIII.

If a right line Z be divided any-wise into two parts, the rectangle comprehended under the whole

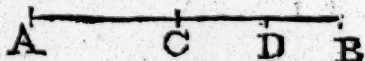


line Z, and one of the segments E four times, together with the square of the other segment A, is equal to the square of the whole line Z, and the segment E, taken as one line Z - E.

I say that $4 ZE + Aq = Q. Z + E.$ For $2 ZE a = a$ 7. 2. and $Zq + Eq - Aq.$ Therefore $4 ZE + Aq = Zq + Eq + 3. ax.$
 $2 ZE b = Q. Z + E.$ Which was to be demonstrated. b 4. 2.

P R O P. IX.

If a right line AB be divided into equal parts



AC, CB, and into unequal parts AD, DB, then are the squares of the unequal parts AD, DB, together, double to the square of the half line AC, and to the square of the difference CD.

I say that $ADq + DBq = 2 ACq + 2 CDq.$ For $ADq + DBq a = ACq + CDq + 2 ACD + DBq.$ But a 4. 2.
 $2 ACD (b \ 2 BCD) + DBq c = CBq (ACq) + CDq.$ b hyp.
d Therefore $ADq + DBq = 2 ACq + 2 CDq.$ Which c 7. 2.
was to be demonstrated. d 2. ax.

This may be otherwise deliver'd and more easily demonstrated thus; The aggregate of the squares made of the sum and the difference of two right lines A, E, is equal to the double of the squares made from those lines,

For $Q: A + E a = Aq + Eq + 2 AE.$ and $Q: A - E a$ 4. 2.
 $b = Aq + Eq - 2 AE.$ These added together make b sch. 72.
 $2 Aq + 2 Eq.$ Which was to be demonstrated.

P R O P.

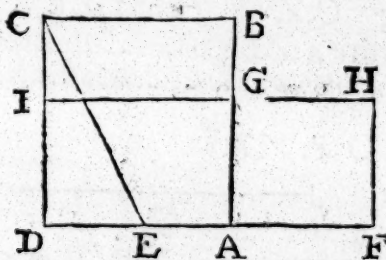
P R O P. X.



If a right line A be divided into two equal parts, and another line be added in a right line with the same, then is the square of the whole line together with the added line (as being one line) together with the square of the added line E , double to the square of $\frac{1}{2} A$, and the added line E , taken as one line.

- a 4. 2. I say that $Eq + Q. A + E$, i. e. $a Aq + 2 Eq + 2$
 b cor. 4. 2. $AE = 2 Q. \frac{1}{2} A + 2 Q. \frac{1}{2} A + E$. For $2 Q. \frac{1}{2} A b =$
 c 4. 2. Aq . And $2 Q. \frac{1}{2} A + E c = \frac{1}{2} Aq + 2 Eq + 2 AE$.
 Which was to be demonstrated.

P R O P. XI.



To cut a right line given AB in a point G , so that the rectangle comprehended under the whole line AB , and one of the segments BG , shall be equal to the square that

is made of the other segments AG .

- a 46. 1. Upon AB describe the square AC . b Bisect the
 b 10. 1. side AD in E , and draw the line EC ; from the line EA produced take $EF = EC$. On AF make the square AH . Then is $AH = AB \times BG$.

- c 6. 2. For HG being drawn out to I ; the rectangle
 d constr. $DH + EAq = EFq = EBq = BAq + EAq$.
 e 47. 1. Therefore is $DH = BAq =$ to the square AC .
 f 3. ax. Take away AI common to both, then remains the square $AH = GC$, that is, $AGq = AB \times EG$. Which was to be done.

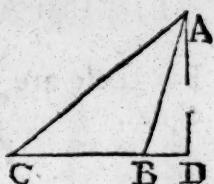
Schol.

- * 6. 13. This proposition cannot be performed by numbers; * for there is no number that can be so divided, that the product of the whole into one part shall be equal to the square of the other part.

P R O P.

P R O P. XII.

In obtuse-angled triangles ABC, the square that is made of the side AC subtending the obtuse angle ABC, is greater than the squares of the sides BC, AB, that contain the obtuse angle ABC, by a double rectangle contained under one of the sides BC, which are about the obtuse angle ABC, on which side produced the perpendicular AD falls, and under the line BD, taken without the triangle from the point on which the perpendicular AD falls to the obtuse angle ABC.



I say that $AC^2 = CB^2 + AB^2 + 2 CB \cdot BD$.

For these are all equal.

$$\begin{cases} a AC^2 \\ b CB^2 + 2 CBD + BD^2 + AD^2 \\ c CB^2 + 2 CBD + AB^2 \end{cases}$$

a 47. 1.
b 4. 2.
c 47. 1.

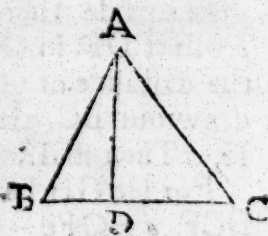
Scholium.

Hence, the sides of any obtuse-angled triangle ABC being known, the segment BD intercepted betwixt the perpendicular AD, and the obtuse angle ABC, as also the perpendicular it self AD, shall be easily found out.

Thus. Let AC be 10, AB 7, CB 5. Then is $AC^2 = 100$, $AB^2 = 49$, $CB^2 = 25$. And $AB^2 + CB^2 = 74$. Take that out of 100, then will 26 remain for $2 CBD$. Wherefore CBD shall be 13; divide this by CB 5, there will $2\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

P R O P. XIII.

In acute angled triangles ABC, the square made of the side AB, subtending the acute angle ACB, is less than the squares made of the sides AC, CB, comprehending the acute angle ACB, by a double rectangle contained under one of the sides BC, which are about the acute angle ACB, on which the perpendicular AD falls, and under the line DC, taken within the triangle from the perpendicular AD, to the acute angle ACB.



I

I say that $ACq + BCq = ABq + 2 BCD$.

a 47. 1.

b 7. 2.

c 47. 1.

For these are
equal.

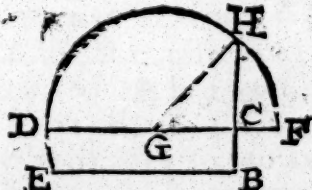
$$\left. \begin{array}{l} ACq + BCq. \\ a \ ADq + DCq + BCq. \\ b \ ADq + BDq + 2 BCD. \\ c \ ABq + 2 BCD. \end{array} \right\}$$

Coroll.

Hence, The sides of an acute-angled triangle ABC being known, you may find out the segment DC, intercepted betwixt the perpendicular AD, and the acute angle ABC, as also the perpendicular it self AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from $ACq + BCq$, that is, from $225 + 196 = 421$. Then remains 252 for $2 BCD$, wherefore BCD will be 126, divide this by BC 14, then will 9 be found out for DC. From whence it follows $AD = \sqrt{225 - 81} = 12$.

P R O P. XIV.



To find a square ML equal to a right lined figure given A.

a 45. 1.

b 10. 1.

a Make the rectangle $DB = A$, and produce the greater side thereof DC to F, so that $CF = CB$, b bisect DF in G, about which as the center at the distance of GF describe the circle FHD, and draw out BC, till it touch the circumference in H. Then shall be $CHq = * ML = A$.

* 46. 1.

c constr.

d 5. 2. and

3. ax.

e 47. 1. and

3. ax.

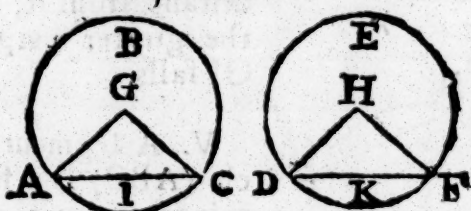
For let GH be drawn. Then is $A c = DB c = DCF d = GFq - GCq e = HCq c = ML$. Which was to be done.

The End of the second Book.

The

The THIRD BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

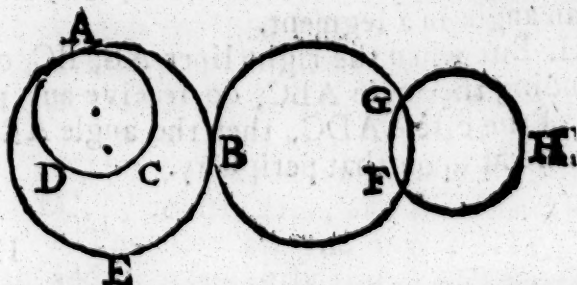


I. **E** Qual circles (GABC, HDEF) are such whose diameters are equal; or, from whose centers right lines drawn HD, are equal.

GA,

II. A right line AB, is said to touch a circle FEDC, when touching the same, and being produced, it cutteth it not.

The right line FG cuts the circle FEDC.

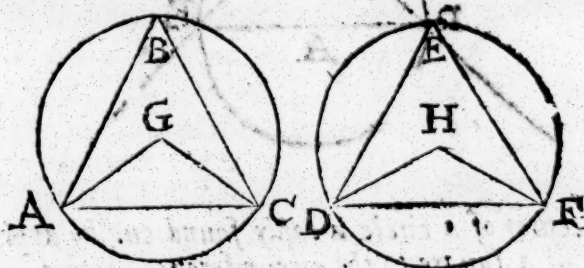
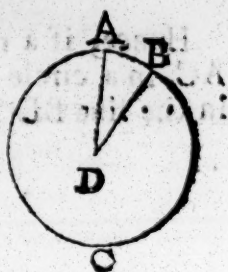


III. Circles DAC, ABE (and also FBG, ABE) are

D

are

IX. A sector of a circle (ADB) is when an angle ADB is set at the center D of that circle; namely, that figure ADB. comprehended under the right lines AD, BD, containing the angle, and the part of the circumference received by them AB.

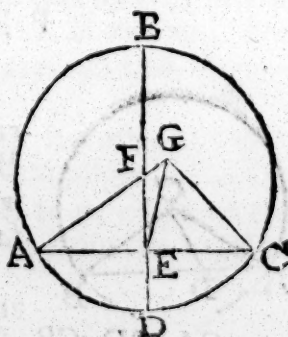


X. Like segments of a circle (ABC, DEF) are those which conclude equal angles (ABC, DEF;) or, in whom the angles ABC, DEF, are equal.

P R O P. I.

To find the center F of a circle given ABC.

Draw a right line AC any-wise in the circle, which bisect in E; thro' E draw a perpendicular DB, and bisect the same in F; the point F shall be the center.



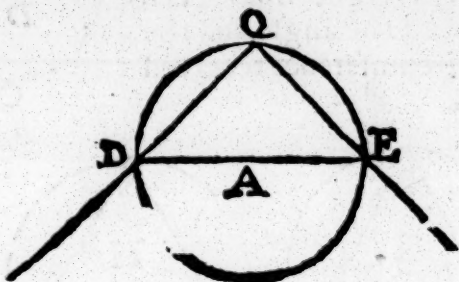
If you deny it, let G, a point without the line BD, be the center (for it cannot be in the line BD, since that cannot be divided equally in any point but F;) let the lines GA, GC, GE, be drawn. Now if G be the center, a then is $GA = GC$, and a 15. def. 1. $AE = EC$, by construction, and the side GE com- b 8. 1. mon. b Therefore are the angles GEA, GEC, c 10. def. 1. equal, and c consequently right. d Therefore the d 12. ax. angle $GEC = FEC$. e Which is absurd. e 9. ax.

D 2

Coroll.

Coroll.

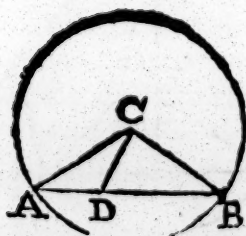
Hence, if a right line BD bisect any right line AC, in a circle at right angles, the center shall be in the line BD that cuts the other.



Andr.
Tacq.

The center of a circle is easily found out by applying the top of a square to the circumference thereof. For if the right line DE that joins the points D, E, in which the sides of the square QD, QE, cut the circumference, be bisected in A, the point A shall be the center. The demonstration whereof depends upon Prop. XXXI. of this Book.

P R O P. II.



If in the circumference of a circle CAB, any two points A, B, be taken, the right line AB, which joins those two parts, shall fall within the circle.

Take in the right line AB any point D; from the center C draw CA, CD, CB. Because CA = CB, therefore is the angle A = B. But the angle CDB < A, therefore is CDB < B, therefore CB < CD. But CB only reaches the circumference, therefore CD comes not so far; wherefore the point D is within the circle. The same may be proved of any other point in the line AB. And therefore the whole line AB falls within the circle. Which was to be dem.

Coroll.

Coroll.

Hence, if a right line touch a circle, so that it cut it not, it touches but in one point.

PROP. III.

If in a circle EABC, a right line BD drawn thro' the center, bisect any other line AG not drawn thro' the center, it shall also cut it at right angles: And if it cuts it at right angles, it shall also bisect the same.



From the center E let the lines EA, EC, be drawn.

1. Hyp. Because $AF = FC$, and $EA = EC$, a hyp. and the side EF common; the angles EFA, EFC, b 15. def. 1. c shall be equal, and d consequently right. Which c 8. 1. was to be demonstrated. d 10. def. 1.

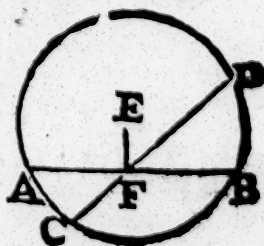
2. Hyp. Because $EFA = EFC$, and the angle e hyp. and $EAF = ECF$, and the side EF common; g there- 12. ax. fore is $AF = FC$. Therefore AC is cut into two f 5. 1. equal parts. Which was to be demonstrated. g 26. 1.

Coroll.

Hence, in any equilateral or Isosceles triangle, if a line drawn from the vertical angle bisect the base, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle bisects the base.

PROP. IV.

If in a circle ACD, two right lines AB, CD, cut thro' one another, yet neither of them pass through the center E, then neither of those lines are divided into equal parts.



For if one line pass thro'

D 3

the

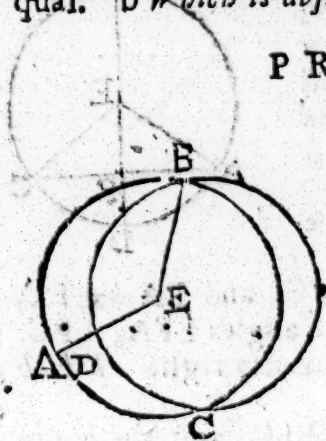
The third Book of

the center, it appears that it cannot be bisected by the other; because by Hypothesis, the other does not pass thro' the center.

If neither of them pass thro' the center, then from the center E draw EF: now if AB, CD, were both bisected in F, then *a* would the angles EFB, EFD, be both right, and consequently equal. *b* Which is absurd.

a 3.3.
b 9. ax.

P R O P. V.

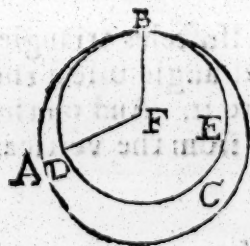


a 15. def. 1.
b 9. ax.

If two circles BAC, BDC, cut one the other, they shall not have the same center E.

For otherwise the lines EB, EDA, drawn from E the common center, would be $DE = EB = EA$. *b* Which is absurd.

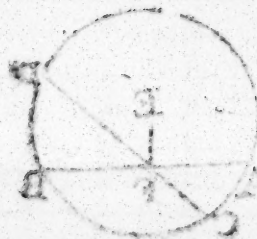
P R O P. VI.



a 15. def. 1.
b 9. ax.

If two circles BAC, BDC, inwardly touch one the other (in B) they have not one and the same center F.

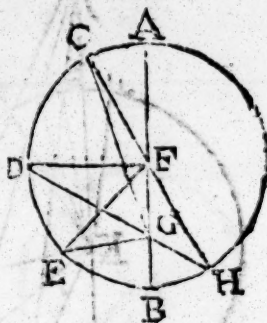
For otherwise the right lines FB, FDA, drawn from the center F, would be $FD = FB = FA$. *b* Which is absurd.



P R O P.

PROP. VII.

If in AB the diameter of a circle some point G be taken, which is not the center of the circle, and from that point certain right lines GC, GD, GE , fall on the circle, the greatest line shall be that (GA) in which is the center F ; the least, the remainder of the same line $(GB.)$ And of all the other



lines, the line GC , nearest to that which was drawn thro' the center is always greater than any line farther removed GD ; and only two lines are equal GE, GH , which fall upon the circle from the same point, on each side of the least GB , or of the greatest GA , and equally distant from the said lines GB, GA .

From the center F draw the right lines FC, FD, FE ; * make the angle $BFH = BFE$. * 23. I.

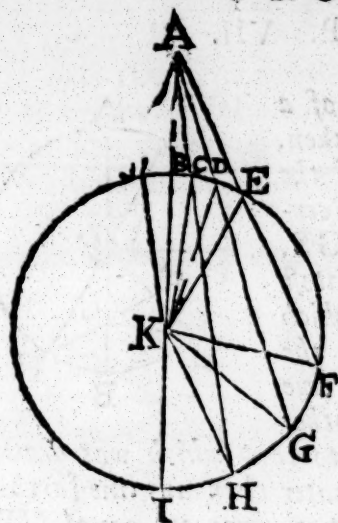
1. $GF + FC$ (that is GA) $a > GC$. Which was a 20. I. to be demonstrated.

2. The side FG is common; and $FC = FD$, and b 15. def. 1. the angle $GFC = GFD$; d wherefore the base c 9. ax. $GC = GD$. d 24. I.

3. $FB (FE) e > GE + GF$. Therefore FG , which e 20. I. is common, being taken away from both, there f 5. ax. remains $BG > EG$.

4. The side FG is common, and $FE = FH$, and the angle $BFH g = BFE$; b Therefore is $GE = GH$. g constr. But that no other line GD from the point G , h 4. I. can be equal to GE , or GH , is already proved.

Which was to be demonstrated.



If some point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF, to the circle, and of those one AI be drawn thro' the center K, and the others anywise; of all those lines that fall on the concave of the circumference, that is the greatest AI, which is drawn thro' the center; and of the others, that which is nearest (AH) to

the line that passes thro' the center, is greater than that which is more distant AG. But of all those lines that fall on the convex part of the circle, the least is that AB, which is drawn from the point A, to the diameter IB; and of the others, that (AC) which is nearest to the least, is less than that which is farther distant AD. And from that point there can be only two equal right lines AC, AL, drawn, which shall fall on the circumference on each side of the least line AB, or of the greatest AI.

From the center K, draw the right lines KH, KG, KF, KC, KD, KE, and make the angle $AKL = AKC$.

a 20. 1.

1. AI (AK + KH) a $\sqsubset$ AH.

b 24. 1.

2. The side AK is common, and $KH = KG$, and the angle $AKH \sqsubset AKG$; b therefore the base $AH \sqsubset AG$.

c 20. 1.

3. KA c $\sqsupset KC + CA$. From hence take away KC, KB, that are equal; then will remain AB, d $\sqsupset AC$.

d 5. ax.

e 21. 1.

4. AC + CK e $\sqsupset AD + DK$. From thence take away CK, DK, that are equal; then remains AC

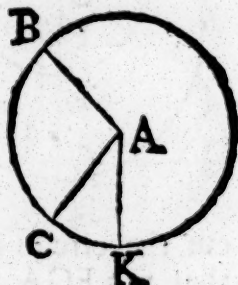
f 5. ax.

f $\sqsupset AD$.

5. The side KA is common, and $KL=KC$ and the angle $AKL=g=AKC$; *h* therefore $LA=CA$. *g* *constr.* But that no other line could be drawn equal to *h* 4. 1. these, was proved above. Therefore, &c.

P R O P. IX.

If in a circle BCK a point A be taken, and from that point more than two equal right lines AB, AC, AK , drawn to the circumference, then is that point A the center of the circle.

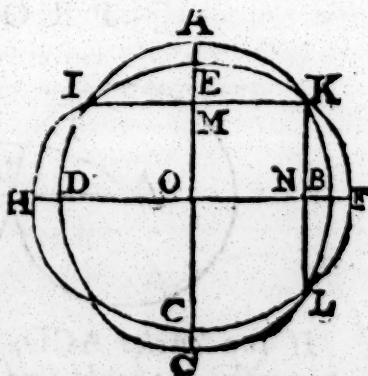


a 7. 3.

For a from no point without the center can more than two right lines equal be drawn to the circumference. Therefore A is the center. Which was to be demonstrated.

P R O P. X.

A circle $IAKBL$ cannot cut another circle $IEKFL$, in more than two points.

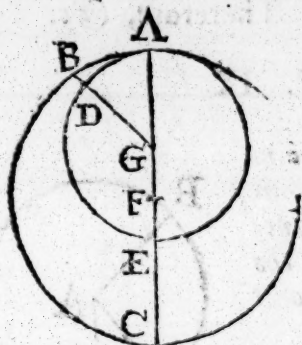


a cor. 1. 3.

Let one circle, if it may be, cut the other in three points I, K, L , and IK, KL , being join'd, let them be bisected in M and N . Both circles have their centers in their perpendiculars MC, NH , and in the intersection of those perpendiculars which is O . Therefore the circles that cut each other have the same center. Which is false, by Prop. 5. 3.

P R O P.

PROP. XI.



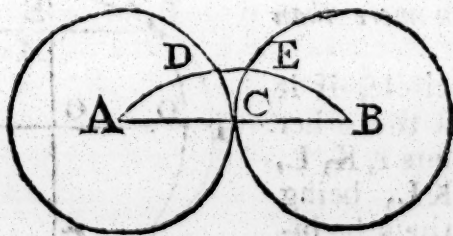
If two circles GADE, FABC, touch one the other inwardly, and their centers be taken G, F; a right line FG joining their centers, and produced, shall cut the circumference in A, the point of contact of the circles.

If it can be, let the right line FG produced cut the circles in some other point than A; so that not FGA, but FGDB, shall be a right line. Let the line GA be drawn. Now, because GD

a 15. def. 1.
b 7. 3.
c 9. ax.

$a = GA$, and $GB \perp GA$ (since the right line FGB passes thro' F, the center of the greater circle) therefore is $GB \perp GD$. c Which is absurd.

PROP. XII.



If two circles ACD, BCE, touch one the other outwardly, the right line AB, which joins their centers A, B, shall pass thro' the point of contact C.

If it may be, let ADEB be a right line cutting the circles, not in the point of contact C, but in the points D, E; draw AC, CB, then is $AD + EB$ ($AC + CB$) $a \sqsubset ADEB$. b Which is absurd.

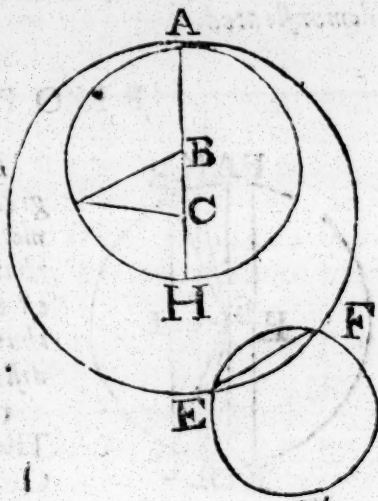
a 20. 1.
b 9. ax.

PROP.

PROP. XIII.

A circle CAF cannot touch a circle BAH in more points than one A, whether it be inwardly or outwardly.

1. Let one circle (if it can be) touch another in two points A, H. Then will the right line CB, that joins the centers, if it be produced, fall as well in A, as H. Now because $CH = CA$, and $BH = CH$, therefore is $BA = BH = CA$. Which is absurd.



a 11. 3.

b 15. def. 1.
c 15. def. 1.
d 9. ax.

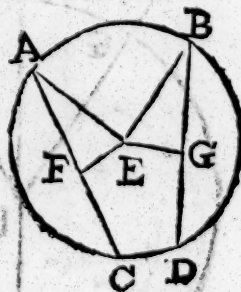
2. If it be said to touch outwardly in the points E and F, then draw the line EF, which will be in both circles. Therefore those circles cut one the other; which is against the Hyp.

e 2. 3.

PROP. XIV.

In a circle EABC, equal right lines AC, BD, are equally distant from the center E: and right lines AC, BD, which are equally distant from the center, are equal among themselves.

From the center E, draw the perpendiculars EF, EG, which will bisect the lines AC, BD. join EA, FB.



a 3. 3.

Hyp. $AC = BD$, therefore $AF = BG$. But b 7. ax. also $EA = EB$; therefore $FEq = EAq - AFq = c 47. 1. EBq - BGq = EGq$. d Therefore $FE = EG$. and 3. ax. 2. Hyp.

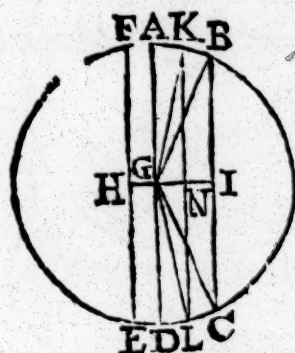
d schol.

48. 1.

6. ax.

2. Hyp. $EF = EG$. Therefore $AFq = EAq - EFq = EBq - EGq = BGq$. Therefore $AF = GB$, and e consequently $AC = BD$. Which was to be demonstrated.

P R O P. XV.



a 15. def. 1.

b 20. 1.

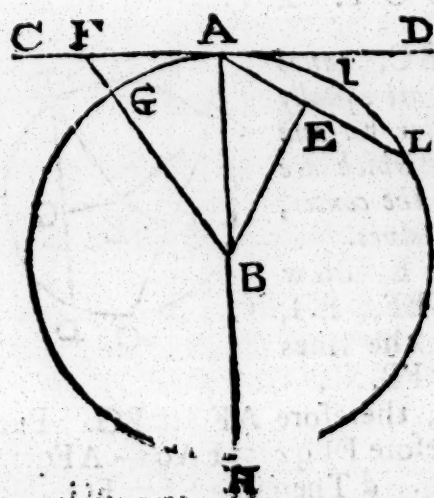
In a circle GABC the greatest line is AD the diameter; and of all other lines, that line FE, which is nearest to the center G, is greater than any line BC farther distant from it.

1. Draw GB, and GC. The diameter AD (a $GB + GC$) $b = BC$.

2. Let the distance GI be $= GH$. Take $GN = GH$. Thro' the point N draw KL perpendicular to GI: join GK, GL. Because $GK = GB$, and $GL = GC$, and the angle $KGL = BGC$; c therefore is $KL (FE) = BC$. Which was to be demonstrated.

c 24. 1.

P R O P. XVI.



A line CD, drawn from the extreme point of the diameter HA, of a circle BALH, perpendicular to the said diameter, shall fall without the circle; and between the same right line and the circumference, cannot be drawn another line AL. And the angle of the

the semicircle BAI, is greater than any right-lined acute angle BAL; and the remaining angle without the circumference DAI, is less than any right-lined angle.

1. From the center B, to any point F, in the right line AC, draw the right line BF. The side BF, subtending the right angle BAF, is a greater a 19. 1. than the side BA, which is opposite to the acute angle BFA. Therefore, whereas BA (BG) reaches to the circumference, BF shall reach further; and so the point F. and for the same reason is any other point of the line AC, placed without the circle.

2. Draw BE perpendicular to AL. The side BA, opposite to the right angle BEA, is b greater than b 19. 1. the side BE, which subtends the acute angle BAE; therefore the point E, and so the whole line EA, falls within the circle.

2. Hence it follows, that any acute angle, to wit, EAD, is greater than the angle of contact DAI. and that any acute angle BAL is less than the angle of a semicircle BAI. Which was to be dem.

Coroll.

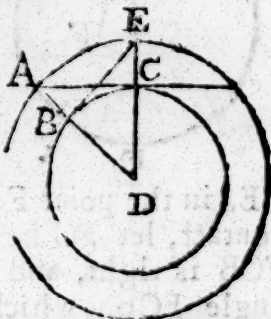
Hence, a right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the said circle.

From this proposition are gathered many paradox and wonderful conjectures, which you may meet with in the interpreters.

P R O P. XVII.

From a point given A to draw a right line AC, which shall touch a circle given DBC.

From D, the center of the circle given. let a line DA, cutting the circumference in B, be drawn to the point given A; from the center D, describe another circle thro' the point A; and from B, draw a perpendicular

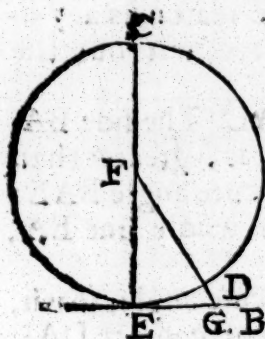


to

to AD, which shall meet with the circle AE in the point E; and draw ED, meeting with the circle BC, in the point C. Then the line drawn from A to C, shall touch the circle DBC.

- a 15. def. 1. For DB $a=DC$, and DE $a=DA$, and the angle D is common; b therefore the angle $ACD=EBD$ and right. c Therefore AC, touches the circle in C. Which was to be done.

P R O P. XVIII.



If any right line AB touch a circle FEDC, and from the center to the point of contact E, a right line FE be drawn; that line FE shall be perpendicular to the tangent AB.

a 2. def. 3.

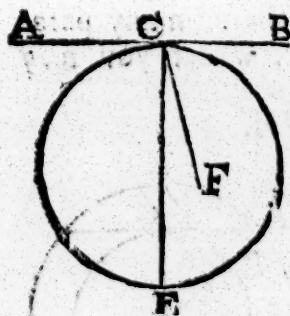
- the circle in D. Therefore, whereas the angle FGE is said to be right, b thence is the angle FEG acute; c so that FE (FD) $\perp$ FG. d Which is absurd.

b cor. 17. 1.

c 19. 1.

d 9. ax.

P R O P. XIX.



If any right line AB touch a circle, and from the point of contact C a right line CE be erected at right angles to the tangent, the center of the circle shall be in the line CE so erected.

a 12. ax.

b 9. ax.

- If you deny it, let the center be without the line CE, in the point F; and from F, to the point of contact, let FC be drawn. Therefore the angle FCB is right, and a consequently equal to the angle ECB, which was right by Hypothesis. b Which is absurd.

P R O P.

PROP. XX.



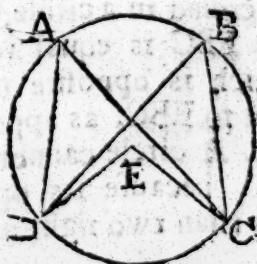
In a circle DABC, the angle BDC at the center is double of the angle BAC at the circumference, when the same arch of the circle BC, is the base of the angles.

Draw the line ADE. The outward angle BDE $a = \text{DAB} + \text{DBA} = 2 \text{ DAB}$. Likewise the angle EDC $b = 2 \text{ DAC}$. Therefore in the first and second cases, the whole angle BDC $= 2 \text{ BAC}$, and in the third case the remaining angle BDC $= 2 \text{ BAC}$. Which was to be demonstrated.

PROP. XXI.

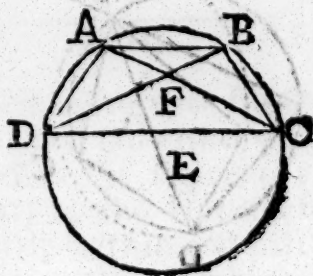
In a circle EDAC, the angles DAC, and DBC, which are in the same segment, are equal one to the other.

1. Case. If the segment DABC be greater than a semicircle, from the center E draw ED, EC. Then is twice the angle A $a = \text{Ea} = 2 \text{ B}$. Which was to be demonstrated.



$a = 2 \text{ B}$.

2. Case. If the segment be less than a semicircle, then is the sum of the angles of the triangle ADF equal to the sum of the angles of the triangle BCF, from each let AFD be taken away, equal to BFC, and ADB

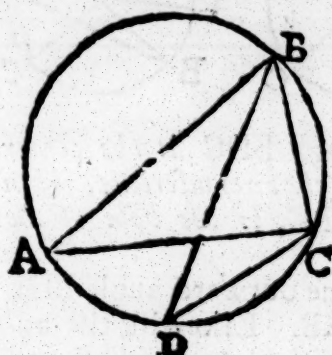


by the 1st case.

$c = \text{ACB}$.

ACB, be likewise taken away, then remains $DAC = DBC$. Which was to be demonstrated.

P R O P. XXII.



The angles ADC, ABC, of a quadrilateral figure ABCD, described in a circle, which are opposite one to the other, are equal to two right angles.

Draw AC, BD. The angle $ABC + BCA + BAC = 2$ right. But $BDA = BCA$, and $BDC = BAC$. There-

fore $ABC + ADC = 2$ right angles. Which was to be demonstrated.

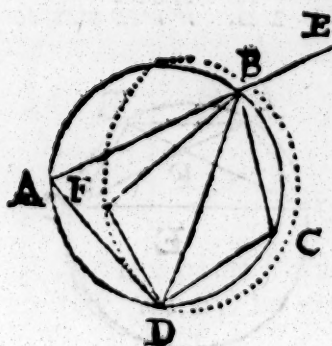
Coroll.

* See the following Diagram.

1. Hence, if one side * AB of a quadrilateral, described in a circle, be produced, the external angle EBC is equal to the internal angle ADC, which is opposite to that ABC, which is adjacent to EBC, as appears by 13. 1. and 3. ax.

2. A circle cannot be described about a Rhombus; because its opposite angles are greater or less than two right angles.

Schol.



If in a quadrilateral ABCD, the angles A, and C, which are opposite, be equal to two right, then a circle may be described about that quadrilateral.

For a circle will pass thro' any three angles (as shall appear by 5. 4.) I say that shall pass thro' A the 4th also of such

a quadrilateral: For if you deny it, let the circle pass through F: Therefore the right lines BF, FD, BD being drawn, the angle $C+F = 2 \text{ right}$ a 22. 3.
 $b = C+A$; wherefore A is equal to F . b hyp.
is absurd. c 3. ax.
d 21. c.

P R O P. XXIII.

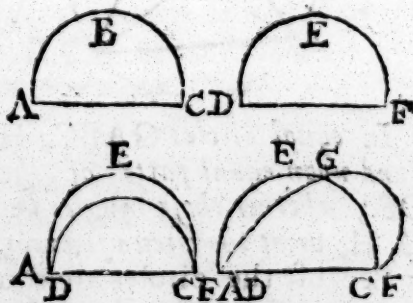
Two like and unequal segments of circles ABC, ADC cannot be set on the same right line AC, and the same side thereof.



For if they are said to be like, draw the line CB cutting the circumferences in D and B, join AB and AD. Because the segments are supposed like, a therefore is the angle $ADC = ABC$. a 10 def. 3.
is absurd. b 16. 1.

P R O P. XXIV.

Like segments, of circles, ABC, DEF upon equal right lines AC, DF, are equal one to the other. The base AC being laid on the base DF will agree with it, because $AC = DF$. There-

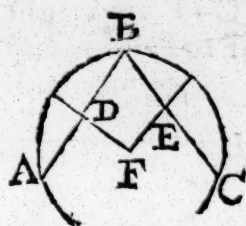


fore the segment ABC shall agree with the segment DEF (for otherwise it shall fall either within or without; and if so a then the segments are not like, which is contrary to the Hypothesis, and at least it shall fall partly within and partly without, and so cut in three points, b which is a 23. 3.
b 10. 3.
c 8. ax.
absurd. c Therefore the segment $ABC = DEF$.
Which was to be demonstrated.

E

P R O.

P R O P. XXV.



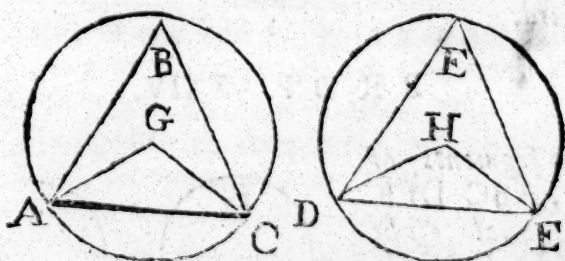
A segment of a circle A-BC being given, to describe the whole circle whereof that is a segment.

Let two right lines be drawn AB, BC, which bisect in the points D and E. From D and E draw the perpendiculars DF,

EF meeting in the point F. I say this point shall be the center of the circle.

a cor. 1.3. For the center shall be as well in a DF as EF, therefore it must be in the common point F. Which was to be done.

P R O P. XXVI.



In equal circles GABC, HDEF, equal angles stand upon equal parts of the circumference, AC, DF; whether those angles be made at the centers G, H, or at the circumferences, B, E.

Because the circles are equal, therefore is GA = HD, and GC = HF; also by Hypothesis the angle G = H; a therefore AC = DF. Moreover the angle B b = $\frac{1}{2}$ G = $\frac{1}{2}$ H b = E. d Therefore the segments ABC, DEF are like, and e consequent-

a 4. 1.

b 20. 3.

c hyp.

d 10 def. 3.

e 24. 3.

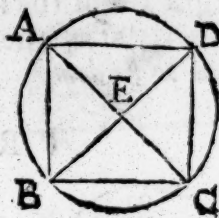
f 3 ax.

ly equal. f whence the remaining segments also AC, DF are equal. Which is to be dem.

Schol.

Schol.

In a circle $ABCD$ let an arch AB be equal to DC ; then shall AD be parallel to BC . For the right line AC being drawn, the angle $ACB = CAD$; wherefore by 27. 1. the said sides are parallel.



a 26. 3.

PROP. XXVII.

In equal circles $GABC$, $HDEF$, the angles standing upon equal parts of the circumference.



AC , DF , are equal between themselves,

whether they be made at the centers G , H , or at the circumferences, B , E .

For if it be possible, let one of the angles AGC be $\neq DHF$, and make $AGI = DHF$; thence is the arch $AI = DF = AC$. Which is absurd.

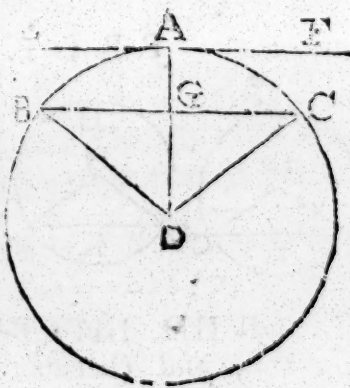
a 26. 3.

b hyp.

c 9. ax.

Schol.

A right line EF , which being drawn from A the middle point of any periphery BC , toucheth the circle, is parallel to the right line BC subtending the said periphery.



From the center D draw a right line DA to the point of contact A , and join DB, DC .

The side DG is common, and $DB = DC$, and the angle $BDA = CDA$, (because the arches $BA = CA$)

a 27. 3.

E 2

CA

angles
AC,
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s also

Schol.

b hyp.

c 4. 1.

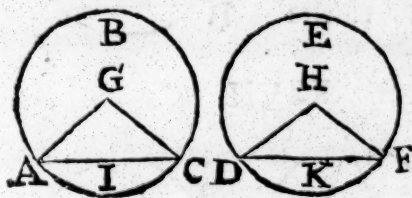
d 10 def. 1.

e hyp.

f 28. 1.

CA are *b* equal) therefore the angles at the base DGB, DGC are *c* equal, and *d* consequently right; but the inward angles GAE, GAF are also *e* right, *f* therefore BC, EF are parallel. Which was to be dem.

P R O P. XXVIII.



In equal circles GA-BC, HDEF, equal right lines AC, DF, cut off equal parts of the circumference, the greatest ABC equal to the greatest DEF,

and the least AIC to the least DKF.

From the centers G, H, draw GA, GC, and HD, HF.

a hyp.

b 8. 1.

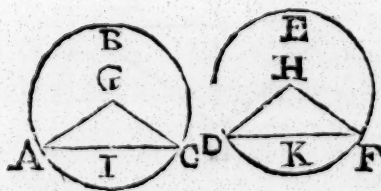
c 26. 3.

d 3. ax.

Because $GA = HD$, and $GC = HF$, and $AC = DF$, *b* therefore is the angle $G = H$; *c* whence the arch $AIC = DKF$; *d* and so the remaining arch $ABC = DEF$. Which was to be dem.

But if the subtended line AC be $\sqsubset$ or $\sqsupset$ than DF, then in like manner will the arch AC be $\sqsubset$ or $\sqsupset$ than DF.

P R O P. XXIX.



In equal circles GA-BC, HDEF, equal right lines, AC, DF subtend equal peripheries ABC, DEF.

Draw the lines GA, GC, and HD, HF. Because $GA = HD$, and $GC = HF$, and (because the arches AC, DF are *a* equal) the angle $G = H$. *c* therefore is the base $AC = DF$. Which was to be dem.

a hyp.

b 27. 3.

c 4. 1.

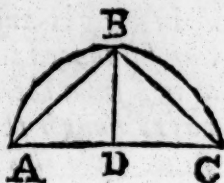
This and the three precedent propositions may be understood also of the same circle.

P R O P.

P R O P. XXX.

To cut a periphery given ABC into two equal parts.

Draw the right line AC, and bisect it in D; from D draw a perpendicular DB meeting with the arch in B, it shall bisect the same.

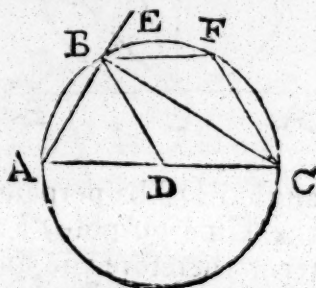


For join AB, and CB. The side DB is common, and $AD = DC$, and the angle $ADB = CDB$. *a constr.* therefore $AB = BC$; *d* whence the arch $AB = BC$. Which was to be done.

b 12. ax.
c 4. 1.
d 28. 3.

P R O P. XXXI.

In a circle the angle ABC, which is in the semi-circle, is a right angle; but the angle, which is in the greater segment BAC, is less than a right angle, and the angle which is in the lesser segment BFC is greater than a right angle.



Moreover, the angle of the greater segment is greater than a right angle, and the angle of the lesser segment is less than a right angle.

From the center D draw DB. Because $DB = DA$, therefore is the angle $A = DBA$, and *a* 5. r. the angle $DCB = DBC$, *b* therefore the angle $BAC = A + ACB = EBC$, *c* so that ABC and EBC are right angles. *W. W. to be dem. e* Therefore BAC is an acute angle. *f W. W. to be dem. e cor. 17. 1.* And further, whereas $BAC + BFC = 2$ right, *f* 22 3. therefore BFC is an obtuse angle. Lastly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB and the periphery of the lesser segment BFC *g* is less than the right angle ABC. Which was to be demonstrated.

g 9 ax.

E 3

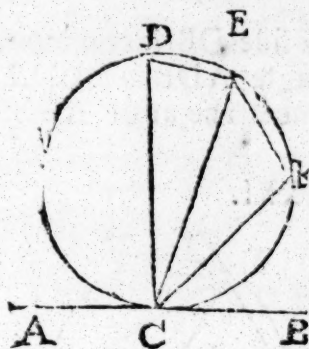
Schol.

The third Book of

Schol.

In a right-angled triangle ABC, if the hypotenuse (or subtended line) AC be bisected in D, a circle drawn from the center D through the point A shall also pass through the point B; as you may easily demonstrate from this prop. and 21. 1.

P R O P. XXXII.

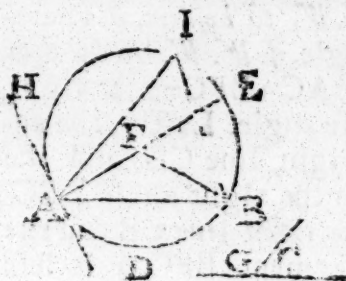


If a right line AB touch a circle, and from the point of contact be drawn a right line CE cutting the circle, the angles ECB, ECA, which it makes with the tangent line, are equal to those angles EDC, EFC which are made in the alternate segments of the circle.

Let CD, the side of the angle EDC be perpendicular to AB (a for it's to the same purpose) b therefore CD is the diameter. c therefore the angle CED in a semicircle is a right angle, d and therefore the angle D + DCE = to a right angle e ECB + DCE. f Therefore the angle D ECB. Which was to be dem.

Now whereas the angle ECB + ECA g = 2 right h = D + F, from both of these take away ECB and D, which are equal, k then remains ECA = F. Which was to be dem.

P R O P. XXXIII.



Upon a right line AB to describe a segment of a circle AIB which shall contain an angle AIB equal to a right lined angle given C.

a Make

a Make the angle $BAD = C$. Through the point A draw the line AE perpendicular to HD . At one end of the line given AB make an angle $ABF = BAF$, let one side thereof cut the line AE in F ; from the center F through the point A , describe a circle which shall pass through B (because the angles $FBA = FAB$, and c therefore $FB = FA$.) AIB is the segment sought. For because HD is perpendicular to the diameter AE , it therefore d touches the circle HD which AB cuts. And therefore the angle $AIB = BAD = C$. Which was to be done.

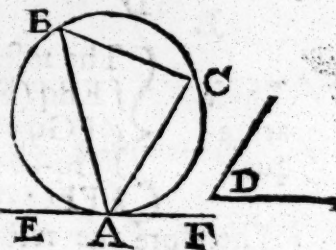
a 23. 1.

b constr.
c 6. 1.

d cor. 16. 3
e 32. 3.
f constr.

P R O P. XXXIV.

From a circle given ABC to cut off a segment ABC containing an angle B equal to a right lined angle given D .



a Draw a right line EF which shall touch the circle given in A . *b* let AC be drawn also making an angle $FAC = D$. This line shall cut off ABC containing an angle $B = CAF = D$. Which was to be done

a 17. 3.
b 23. 1.
c 32. 3.
d constr.

P R O P. XXXV.

If in a circle $DBCA$ two right lines AB , DC cut each other, the rectangle comprehended under the segments AE , EB of the one, shall be equal to the rectangle comprehended under the segments CE , ED of the other.

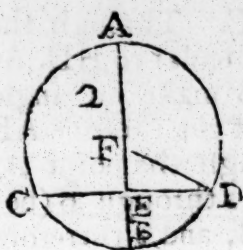


1. Case. If the right lines cut one the other in the center, the thing is evident.

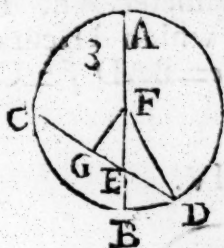
E 4

2. Case

a 5. 2.
b fch. 48. 1.
c 47. 1.
d hyp.
e 3. ax.



2. Case. If one line AB pass thro' the center F, and bisect the other line CD, then draw FD. Now the rectangle AEB + FEQ $a = FBq$ $b = FDq$ $c = EDq + FEQd = CED + FEQ$. Therefore the rectangle AEB = CED. Which was to be dem.



3. Case. If one of the lines AB be the diameter, and cut the other line CD unequally, bisect CD by FG a perpendicular from the center.

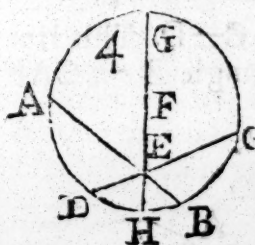
f 5. 2.
g 47. 1.
h 5. 2.
k 47. 1.

These
are e-
qual.

{ The rectangle AEB + FEQ.
f FBq (FDq.)
g FGq + GDq.
FGq + h GEQ + Rectang. CED.
k FEQ + CED.

l 3. ax.

l Therefore the rectangle AEB = CED.

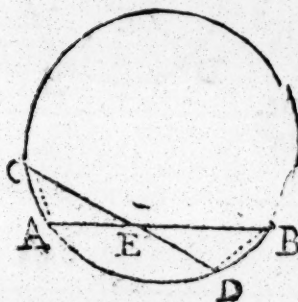


4. Case. If neither of the right lines AB, CD pass thro' the center, then through the point of intersection E, draw the diameter GH. By that which hath been already demonstrated, it appears that the rectangle AEB = GEH = CED. Which was

to be demonstrated.

a 15. 1.
b 21. 3.

c cor. 32. 1.
d 4. 6.
e 16. 6.



demonstrated.

More easily, and generally, thus; join AC and BD, then because the angles a CEA, DEB, and b also C, B (upon the same arch AD) are equal, thence are the triangles CEA, BED c equiangular. d Wherefore CE, EA :: EB, ED. and e consequently CE x ED = AE x EB. W. w. to be

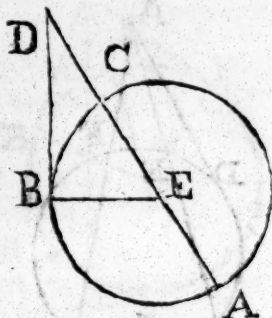
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The citations out of the 6. Book, both here and in the following *prop.* have no dependance on the same ; so that it was free to use them.

P R O P. XXXVI.

If any point be taken without a circle EBC, and from that point two right lines DA, DB fall upon the circle, whereof one DA cuts the circle, the other DB touches it, the rectangle comprehended under the whole line DA that cuts the circle, and DC, that part, which is taken from the point given D to the convex of the periphery, shall be equal to the square made of the tangent line.

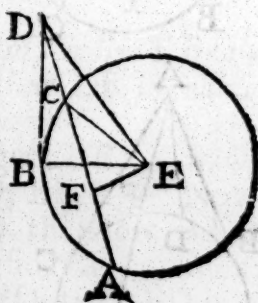
1. *Case.* If the secant AD pass thro' the center, then join EB, this *a* will make a right angle with the line DB, wherefore $DBq + dEBq(EBq) b = EDqc = AD \times DC + ECq$. Therefore $AD \times DC = DBq$. Which was to be demonstrated.



a 18. 3.
b 47. 1.
c 6. 2.
d 3. ax.

2. *Case.* But if AD pass not thro' the center, then draw EC, EB, ED, and EF perpendicular to AD, *a* wherefore AC is bisected into F.

Because $BDq + EBq b = DEq$
 $b = EFq + FDqc = EFq + ADC + FCq$
 $d = ADC + CEq (EBq.) e$
Therefore is $BDq = ADC$.
Which was to be demonstrated.



a 3. 3.
b 47. 1.
c 6. 2.
d 47. 1.
e 3. ax.

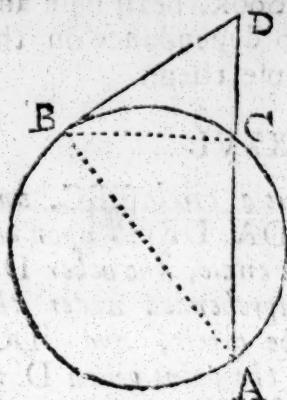
More

232.3.

b 32. 1.

с 4. 6.

d 17.6.



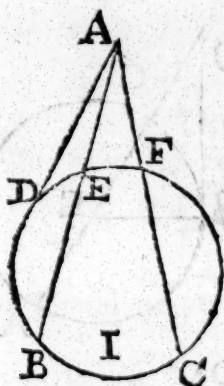
More easily, and generally thus; draw AB and BC. Then because the angles A, and DBC *a* are equal, and the angle D common to both, thence are the triangles BDC, ADB *b* equiangular. *c* Wherefore AD, DB, DB, CD and *d* consequently $AD \times DC = DB^2$. Which was to be demonstrated.

Coroll.

1. Hence, If from any point A taken without a circle, there be several lines AB, AC drawn which cut the circle, the rectangles comprehended under the whole lines AB, AC and the outward parts AE, AF are equal between themselves.

For if the tangent AD be drawn, then is $CAF = ADq$ and $= BAE$.

a 36. 3.

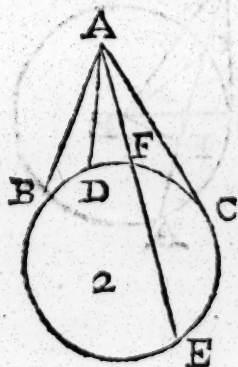


2. It appears also from hence, that if two lines AB, AC drawn from the same point do touch a circle, those two lines are equal one to the other.

For if AE be drawn cutting the circle, then is $ABq a = EAF b = ACq$.

३३८ ३०

b36.3.



3. It is also evident that from a point A taken without a circle, there can be drawn but two lines AB, AC that shall touch the circle.

For

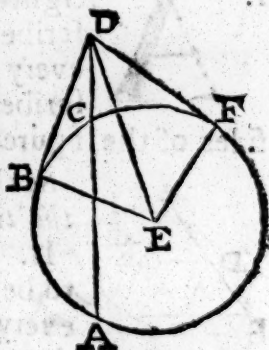
For if a third line AD be said to touch the circles, thence is $AD = AB = AC$. Which is absurd.

4. And on the contrary, it is plain, that if two equal right lines, AB, AC fall from any point A upon the convex periphery of a circle, and that if one of these equal lines AB touch the circle, then the other AC touches the circle also.

For if possible, let not AC, but another line AD, touch the circle; therefore is $AD = AC = AB$. Which is absurd.

P R O P. XXXVII.

If without a circle EBF any point D be taken, and from that point two right lines DA, DB fall on the circle, whereof one line DA cuts the circle, the other DB falls upon it; and if also the rect-angle comprehended under the whole line that cuts the circle, and under that part of it DC which is taken betwixt the point D and the convex periphery, be equal to that square which is made of the line DB falling on the circle, I say that that line DB so falling shall touch the circle given.



From the point D let a tangent DF be drawn, and from the center E draw ED, EB, EF. Now because $DB^2 = DC \cdot DA$, therefore is $DB = DF$: But $EB = EF$, and the side ED common; therefore the angle $EBD = EFD$; but EFD is a right angle, and therefore EBD is right also; and therefore DB touches the circle. Which was to be dem.

Coroll.

From hence it follows that the angle $EDB = EDF$.

The End of the third Book,

THE

The FOURTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

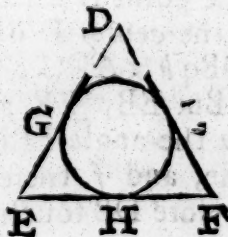
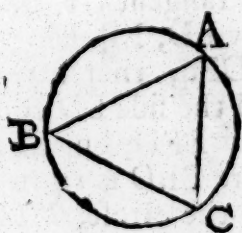
- I. **A** Right-lined figure is said to be inscribed in a right-lined figure, when every one of the angles of the inscribed figure touch every one of the sides of the figure wherein it is inscribed.



So the triangle DEF is inscribed in the triangle ABC.

- II. In like manner a figure is said to be described about a figure, when every one of the sides of the figure circumscribed touch every one of the angles of the figure about which it is circumscribed.

So the triangle ABC is described about the triangle DEF.



- III. A right-lined figure is said to be inscribed in a circle, when all the angles of that figure which is inscribed do touch the circumference of the circle.

- IV. A right-lined figure is said to be described about a circle, when all the sides of the figure which

which is circumscribed touch the periphery of the circle.

V. After the like manner a circle is said to be inscribed in a right-lined figure, when the periphery of the circle touches all the sides of the figure, in which it is inscribed.

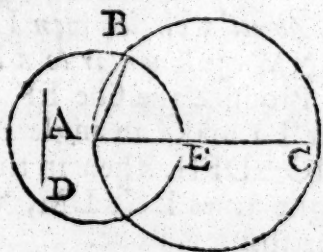
VI. A circle is said to be described about a figure when the periphery of the circle touches all the angles of the figure, which it circumscribes.

VII. A right-line is said to be coapted or applied in a circle when the extremes thereof fall upon the circumference; as the right line AB.



PROP. I. Probl. 1.

In a circle given ABC to apply a right-line AB equal to a right line given D, which doth not exceed AC the diameter of the circle.

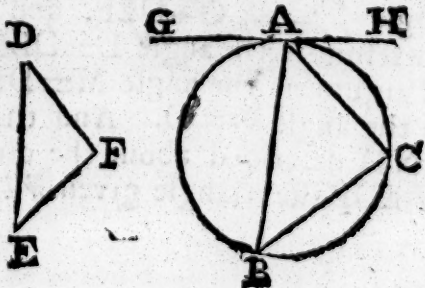


From the center A by the space of $AE = D$ describe a circle meeting with the circle given in B, draw AB. Then is $AB = AE = D$. Which was to be done.

23. post. &
3. 1.
b 15. def. 1.
c constr.

PROP. II. Probl. 2.

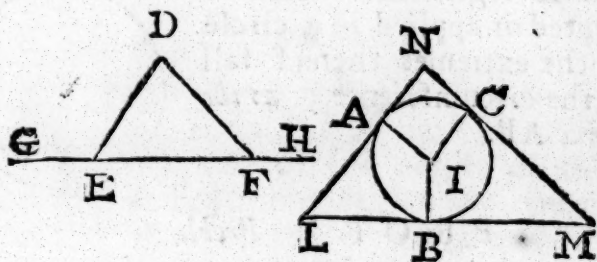
In a circle given ABC to describe a triangle ABC, equiangular to a triangle given DEF.



Let the right line

- a 17. 3. line GH *a* touch the circle given in A; *b*
 b 23. 1. make the angle $HAC = E$, *b* and the angle $GAB =$
 c 32. 3. F. then join BC; and the thing is done.
 d *constr.* For the angle $Bc = HAC$ $d = E$, and the angle
 e 32. 1. $Cc = GAB$ $d = F$; *e* whence also the angle BAC
 $= D$. Therefore the triangle BAC inscribed in
 the circle is equiangular to DEF. *Which was to be*
done.

P R O P. III.



About a circle given IABC to describe a triangle LNM equiangular to a triangle given DEF.

- a 23. 1. Produce the side EF on both sides; at the cen-
 ter I *a* make an angle $AIB = DEG$, and an angle
 b 17. 3. $BIC = DFH$. Then in the points A, B, C let three
 right lines LN, LM, NM *b* touch the circle, and
 the thing is done.

- c 13. *ax.* For it's evident that the right lines LN, LM,
 d 11. 3. MN will meet and make a triangle, *c* because the
 angles LAI, LBI are right; so that the *d* right
 line AB produced will make the angles LAB, L-
 BA, less than two right angles.

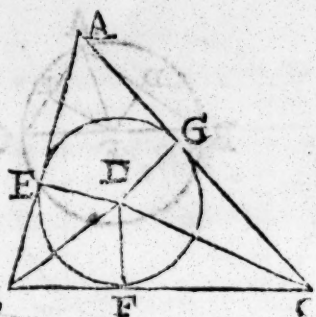
- e *sch.* 32. 1. Since therefore the angle $AIB + Le = 2$ right
 f 13. *ax.* angles $f = DEG + DEF$, and AIB $g = DEG$; *b*
 g *constr.* therefore is the angle $L = DEF$. By the like way
 h 3. *ax.* of argument the angle $M = DEF$. *k* Therefore al-
 k 32. 1. so the angle $N = D$. And therefore the triangle
 LNM described about the circle is equiangular
 to EDF the triangle given. *Which was to be done.*

P R O P.

PROP. IV.

In a triangle given A-BC, to describe a circle EFG.

a Bisect the angle B and C with the right lines BD, CD meeting in the point D, b and draw the perpendicular DE, DF, DG. A circle B described from the center D through E, will pass through G and F,



a 9. 1.

b 12. 1.

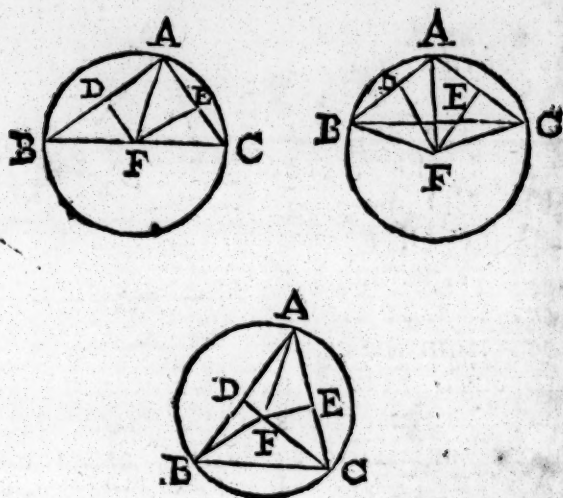
and touch the three sides of the triangle. For the angle DBE $c = DBF$; and the angle DEB $d = DFB$; and the side DB common. ^{c constr.} ^{d 12. ax.} therefore $DE = DF$. By the like argument $DG = DF$. ^{e 26. 1.} The circle therefore described from the center D passes through the 3 points E, F, G. and whereas the angles at E, F, G are right, therefore it touches all the sides of the triangle. Which was to be done.

Schol.

Hence, The sides of a triangle being known, their segments which are made by the touchings of the circle inscribed shall be found, Thus;

Let AB be 12, AC 18, BC 16, then is $AB + BC = 28$. Out of which subduct $18 = AC = AE + FC$, then remains $10 = BE + BF$. Therefore BE, or $BF = 5$; and consequently FC, or CG, $= 11$. Wherefore GA, or AE, $= 7$.

PROP.



About a triangle given ABC to describe a circle FABC.

a 10. &
11. 1.

a Bisect any two sides BA, CA with perpendiculars DF, EF, meeting in the point F. I say this shall be the center of the circle.

b constr.
c constr. &
12. ax.
d 4 1.

For, let the right lines FA, FB, FC be drawn. Now because AD $\equiv$ DB and the side DF common, and the angles FDA $\equiv$ FDB, therefore is FB $\equiv$ FA. After the same manner is FC $\equiv$ FA. Therefore a circle described from the center F shall pass thro' the angles of the triangle given (*viz*) B, A, C, Which was to be done.

Coroll.

* 31. 3.

* Hence, if a triangle be acute-angled, the center shall fall within the triangle; if right-angled, in the side opposite to the right angle, and if obtuse-angled, without the triangle.

Schol.

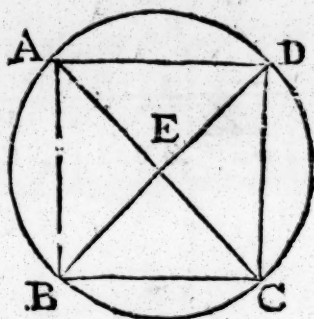
By the same method may a circle be described, that shall pass through 3 points given, not being in the same strait line,

P R O P.

PROP. VI.

In a circle given
EABCD to inscribe a
square ABCD.

a Draw the diameters
AC, BD cutting each
other at right angles in
the center E. Join the
extremes of these diame-
ters with the right lines
AB, BC, CD, DA. And
the thing is done.



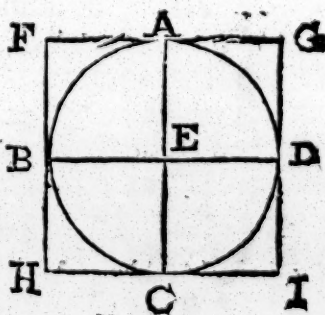
a II. 1.

Now because the four angles at E are right, the
b arches and c subtended lines AB, BC, CD, DA b 26. 3.
are equal; therefore is the figure ABCD equi- c 29. 3.
lateral, and all the angles in semicircles, and so
d right. e Therefore ABCD is a square inscri- d 31. 3.
bed in a circle given. Which was to be done. e 29. 3.
def. 1.

PROP. VII.

About a circle given
EABCD to describe a
square FHIG.

Draw the Diameters
AC, BD cutting one the
other at right angles;
through the extremes of
these diameters a draw
tangents meeting in F, H,
I, G, then I say it's done.



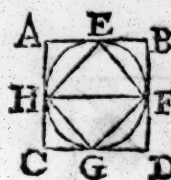
a 17. 3.

For because b the angles A and C are right, c b 18. 3.
therefore is FG parallel to HI. After the same c 28. 1.
manner is FH parallel to GI, and therefore FHIG
is a Pgr. and also right-angled. It is equilateral
because FG d = HI d = DB e = CA d = FH d = d 34. 1
GI. Wherefore FHIG is a f square described e 15. def. 1.
about the circle given. Which was to be done. f 29. def. 1.

F

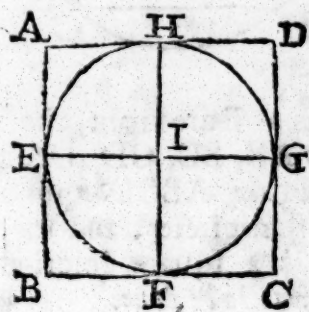
Schol.

Schol.



A square ABCD described about a circle is double of the square EF-GH inscribed in the same circle. For the rectangle $HB = 2$ HEF and $HD = 2$ HGF by the 41. 1.

P R O P. VIII.



In a square given ABCD to inscribe a circle IEFGH.

Bisect the sides of the square in the points H, E, F, G, cutting one the other in I. a circle drawn from the center I through H shall be inscribed in the square.

a 7. ax.
b hyp.
c 33. 1.

d 7. ax.
e 34. 1.

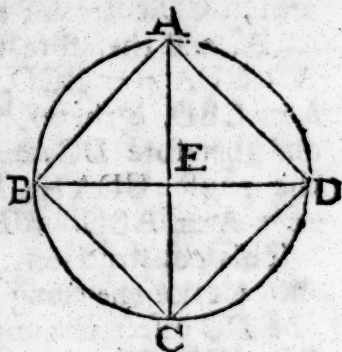
For because AH and BF are *a* equal and *b* parallel, *c* therefore is AB parallel to HF parallel to DC. After the same manner is AD parallel to EG, parallel to BC; therefore IA, ID, IB, IC are parallelograms. Therefore $AH = AE$ *d* $= HI = EI = FI = IG$. The circle therefore described from the center I through H shall pass through H, E, F, G, and touch the sides of the square being the angles H, E, F, G are right. Which was to be done.

P R O P.

P R O P. IX.

About a square given ABCD to describe a circle EABCD.

Draw the diagonals AC, BD cutting one the other in E. From the center E through A describe a circle, then I say that circle is described about the square.

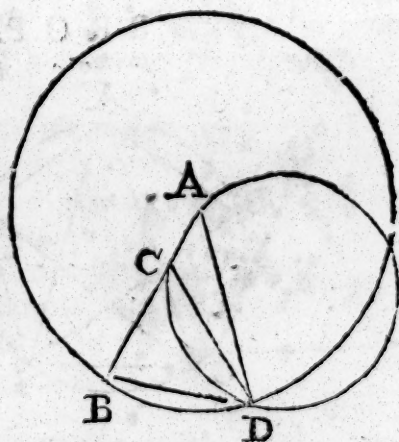


For the angles ABD and BAC are a half of a 4. Cor. right angles, therefore $EA = EB$. After the 33. 1. same manner is $EA = ED = EC$. The circle b 6. 1. therefore described from the center E passes through A, B, C, D the angles of the square given. Which was to be done.

P R O P. X.

To make an Isosceles triangle ABD, having each angle at the base B and ADB double to the remaining angle A.

Take any right line AB, and divide it in C, a so that $AB \times BC$ may be equal to AC^2 . From the center A through B, describe the circle ABD; and in this circle b apply $BD = AC$, and join AD; I say b 1. 4. ABD is the triangle required.



a 11. 2.

F 2

For

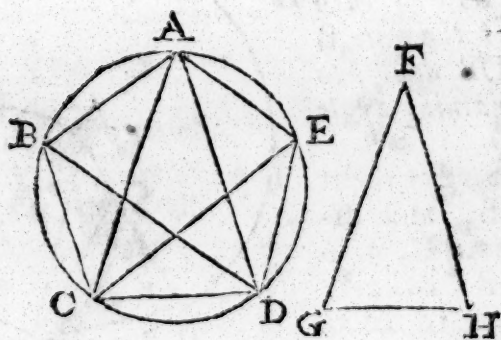
For draw DC, and through the points C, D, A,
 c draw a circle. Now because $AB \times BC = ACq$,
 d it is evident that BD touches the circle ACD
 which CD cutteth; e therefore is the angle BDC
 = A, and therefore the angle BDC + CDA f =
 A + CDA g = BCD. But BDC + CDA = BDA
 h = CBD. k therefore the angle BCD = CBD,
 and therefore DC l = DB = m AC, n wherefore
 the angle CDA = A = BDC. therefore ADB
 = 2 A = ABD. Which was to be done.
 m constr.
 n 5. 1

This construction is Analytically found out
 thus; take the thing for done, and let the right
 line DC bisect the angle BDA; a therefore DA,
 DB :: CA, CB. also because the angle CDA b =
 $\frac{1}{2}$ ADB c = A, c therefore CA = DC. and be-
 cause the angle DCB b = A + CDA = 2 A c =
 B, d thence will be DB = DC. f from whence
 also DB = CA. and so DA (BA.) CA :: CA, CB. g
 whence $BA \times CB = CAq$.

Coroll.

Whereas all the angles A, B, D h make up two
 right angles, it's evident that A is $\frac{1}{5}$ of two right
 angles.

PROP. XI.



In a circle given ABCDE to describe a Pentagon
 figure ABCDE equilateral and equiangular.

a 10. 4.
 b 2. 4. a Describe an Isosceles triangle FGH, having
 each angle at the base double to the other; b in-
 scribe

scribe a triangle CAD equiangular to the said triangle FGH. *c* Bisect the angles at the base ACD and ADC with the right lines DB, CE meeting with the circumference in B and E. join the right lines CB, BA, AE, ED. Then I say it is done.

For it is evident by construction, that the angles CAD, CDB, BDA, DCE, ECA are equal; wherefore the *d* arches and *e* subtended lines DC, CB, BA, AE, DE are equal. Therefore the Pentagon is equilateral, and equiangular *f* because the angles of it BAE, AED, &c. stand on equal *g* arches BCDE, ABCD, &c.

d 26. 3.
e 29. 3.
f 27. 3.
g 2. ar.

A more easy practice of this problem shall be deliver'd at 10. 13.

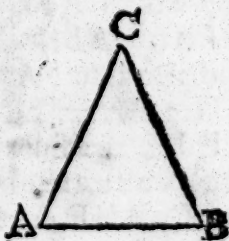
Coroll.

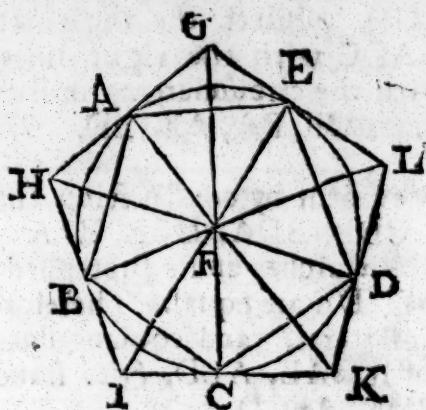
Hence, each angle of an equilateral and equiangular Pentagon is equal to $\frac{2}{3}$ of two right angles or $\frac{2}{5}$ of one right angle.

Schol.

Generally all figures of odd number of sides are inscribed in circles by the help of Isosceles triangles, whose angles at the base are multiples of those at the top: and figures of even number of sides are inscribed in a circle by the help of Isosceles triangles, whose angles at the base are multiples sesquialter of those at the top. *Pet. Herig.*

As in the Isosceles triangle CAB if the angle $A = 3 C = B$. then will AB be the side of a Heptagon. If $A = 4 C$; then is AB the side of an Enneagone. But if $A = \frac{1}{2} C$ then is AB the side of a square. And if $A = 2 \frac{1}{2} C$ CAB will subtend the sixth part of a circumference and likewise if $A = 3 \frac{1}{2} C$ then will AB be the side of an Octagone.





About a circle given FABCDE, to describe an equilateral and an equiangular pentagon HIKLG.

a 11. 4.

b cor. 16.

13.

c 2. cor.

36. 3.

d 8. 1.

e 27. 3.

f 7. ax.

g 12. ax.

h 26. 1.

k 2. ax.

a Inscribe a pentagon ABCDE in the circle given; and from the center draw the right lines FA, FB, FC, FD, FE; and to those lines draw so many perpendiculars GAH, HBI, ICK, KDL, LEG, meeting in the points H, I, K, L, G. then I say it is done. For because GA, GE from the same point G b touch the circle, c therefore is $GA = GE$, and d therefore the angle $GFA = GFE$, therefore the angle $AFE = 2 GFA$. After the same manner is the angle $AFH = FHB$, and consequently the angle $AFB = 2 AFH$. e But the angle $AFE = AFB$, f therefore the angle $GFA = AFH$. But also the angle $FAH g = FAG$, and the side FA is common, h therefore $HA = AG = GE = EL$, &c. k Therefore HG, GL, LK, KI, IH the sides of the pentagon are equal, the angles also are, because double of the equal angles AGF, AHF. therefore, &c.

Coroll.

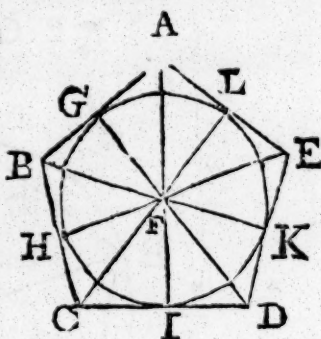
After the same manner, if any equilateral and equiangular figure be described in a circle, and at the extreme points of the semi-diameters drawn from the center at angles, be drawn perpendicular lines to the said diameters, I say that these per-

perpendiculars shall make another figure of as many equal sides and equal angles, described about the circle.

P R O P. XIII.

In an equilateral and equiangular pentagon given ABCDE to inscribe a circle FGHK.

a Bisect two angles of the pentagon A and B with the right lines AI, BK meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle described from the center F through G will touch all the sides of the pentagon.



a 9. 1.

Draw FC, FD, FE. Because BA ^b = BC and the side BF common, and the angle FBA ^c = FBC, ^d therefore is AF = FC and the angle FAB = FCB, but the angle FAB ^e = $\frac{1}{2}$ BAE = $\frac{1}{2}$ BCD. Therefore the angle FCB = $\frac{1}{2}$ BCD. After the same manner are all the whole angles C, D, E bisected. Now whereas the angle FGB ^f = FHB, and the angle FBH = FBG and the side FB is common, ^g therefore is FG = FH. In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle described from the center F through G passes through the points H, I, K, L and ^h touches the side of the pentagon because the angles at those points are right. Which was to be done.

Coroll.

Hence, If any two nearest angles of an equilateral and equiangular figure, and from that point in which the lines meet that bisect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure shall be bisected.

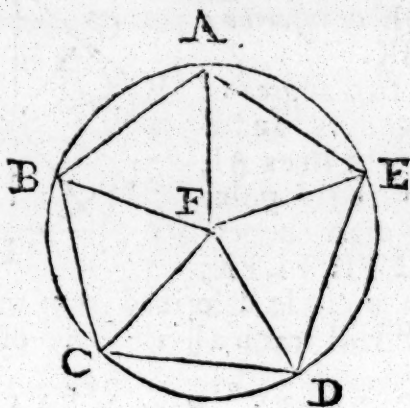
F 4

Schol.

Schol.

By the same method shall a circle be inscribed in any equilateral and equiangular figure.

P R O P. XIV.



About a pentagon given ABCDE equilateral and equiangular to describe a circle FABCDE.

Bisect any two angles of the pentagon with the right lines AF, BF meeting in the point F; the circle described from the center F through A shall be described about the pentagon.

a cor. 13. 4.
b 6. 1.

For let FC, FD, FE be drawn. *a* Then the angles C, D, E are bisected; *b* and therefore FA, FB, FC, FD, FE are equal; therefore the circle described from the center F passes through A, B, C, D, E all the angles of the pentagon. Which was to be done.

Schol.

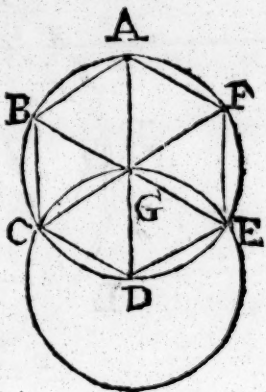
By the same method is a circle described about any figure which is equilateral and equiangular.

PROP.

P R O P. XV.

In a circle given GABCD-
EF to inscribe an Hexagone
(or six-sided figure) equilateral
and equiangular ABCDEF.

Draw the diameter AD; from the center D through the center G describe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. Then I say it's done.



For the angle CGD $a = \frac{1}{2}$ of 2 right $a = DGE$ a 32. 1.
 $b = AGF$ $b = AGB$. c Therefore $BGC = \frac{1}{2}$ of b 15. 1.
 2 right $= FGE$; therefore the d arches and e sub- c cor. 13. 1.
 tentes AB, BC, CD, DE, EF are equal. There- d 26. 3.
 fore the Hexagon is equilateral; but it is equi- e 29. 3.
 angled also, f because all the angles of it stand f 27. 3.
 upon equal arches.

Coroll.

1. Hence, The side of an Hexagon inscribed in a circle is equal to the semidiameter.

2. Hereby an equilateral triangle ACE may very easily be described in a circle given,

Schol.

To make a true Hexagon upon a right line given *Andr.*
 CD. *Tacq.*

a Make an equilateral triangle CGD upon the a 1. 1.
 line given CD; from the center G though C and
 D describe a circle. That circle shall contain the
 Hexagon made upon the given line CD.

P R O P.

The FIFTH BOOK
OF
EUCLIDE'S
ELEMENTS.

Definitions.

I. **A** Part, is a magnitude of a magnitude, a less of a greater, when the less measureth the greater.

II. Multiplex is a greater magnitude in respect of a lesser, when the lesser measureth the greater.

III. Ratio (or rate) is the mutual habitude or respect of two magnitudes of the same kind each to other, according to quantity.

In every ratio that quantity which is referr'd to another quantity is called the antecedent of the ratio, and that to which the other is referr'd is called the consequent of the ratio. as in the ratio of 6 to 4, 6 is the antecedent and 4 the consequent

Note, The quantity of any ratio is known by dividing the antecedent by the consequent ; as the ratio of 12 to 5 is expressed by $\frac{12}{5}$; or the quantity of the

ratio of A to B is $\frac{A}{B}$. Wherefore, often for brevity

for sake we denote the quantities of ratio's thus ; $\frac{A}{B}$

$\rhd$, or $=$, or $\lhd$ $\frac{C}{D}$. that is, the ratio of A to B is greater, equal, or less than the ratio of C to D. And this note must be diligently observed in the understanding of the following Book.

Con.

The fifth Book of

Concerning the divers species of ratio's, you may please to consult interpreters.

IV. Proportion is a similitude of ratio's

That which is here termed proportion, is more rightly called proportionality or analogy; for proportion commonly denotes no more than the ratio betwixt two magnitudes.

V. Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other.

E, 12 A, 4. B, 6. G, 24. VI. Magnitudes are F, 30. C, 10. D, 15. H, 60. said to be in the same ratio, the first A to the

second B, and the third C to the fourth D, when the equimultiples E and F of the first A, and the third C compared with the equimultiples G, H of the second B and the fourth D, according to any multiplication whatsoever, either both together E, F are less than GH both together, or equal taken together, or exceed one the other together, if those be taken E, G and F, H, which answer one to the other.

The note hereof is ::; as A. B :: C. D. That is, as A is to B, so is C to D. which signifies that A to B, and C to D, are in the same ratio. We sometimes

thus express it; $\frac{A}{C} = \frac{B}{D}$, that is, A. B :: C. D.

VII. Magnitudes that have the same ratio (A. B :: C. D. are called proportional.

E, 30. A, 6. B, 4. G, 28. VIII. When of equimultiples, E the

multiplex of the first magnitude A exceeds G the multiplex of the second B, but F the multiplex of the third C exceeds not H the multiplex of the fourth D, then the first A to the second B has a greater ratio than the third C to the fourth D.

If $\frac{A}{B} < \frac{C}{D}$, it is not necessary from this definition

that

that E should always exceed G, when F is less than H; but it is granted that this may be.

IX. Proportionality consists in three terms at least. Whereof the second supplies the place of two.

X. When 3 magnitudes A, B, C are proportional, the first A shall have a duplicate ratio to the 3 C of that it hath to the second B: But when four magnitudes A, B, C, D are proportional, the first A shall have a triplicate ratio to the fourth D of what it had to the second B; and so always in order one more, as the proportion shall be extended.

Duplicate ratio is thus expressed $\frac{A}{C} = \frac{A}{B}$ twice, that is, the ratio of A to C is double of the ratio of A to B.

Triple ratio is thus expressed; $\frac{A}{D} = \frac{A}{B}$ thrice. That is, the ratio of A to D is triple of the ratio of A to B.

$\therefore$ denotes continued proportionals; as A, B, C, D; or 2, 6, 18, 64. are $\therefore$

XI. Magnitudes of a like ratio, are antecedents to antecedents, and consequents to consequents; As if A. B :: C. D. A and C; and B and D are homologous or magnitudes of a like ratio.

XII. Alternate proportion is the comparing of antecedent to antecedent, and consequent to consequent. As if A. B :: C. D. therefore alternately, or by permutation, A. C :: B. D. by the 16. of 5.

In this definition, and the 5 following, names are given to the six ways of arguing which are often used by Mathematicians: the force of which inferences depends on the propositions of this book, which are named in their explications.

XIII. Inverse ratio is when the consequent is taken as the antecedent, and so compared to the antecedent as the consequent; as A. B :: C. D. therefore generally B. A :: D. C. by cor. 4. 5.

XIV. Compounded ratio is when the antecedent and consequent taken both as one are compared to the

the consequent it self. *As* $A. B :: C. D.$ therefore by composition $A + B. B :: C + D. D$ by 18. 5.

XV. Divided ratio is when the excess wherein the antecedent exceedeth the consequent, is compared to the consequent. *As* $A. B :: C. D.$ therefore by division $A - B. B :: C - D. D$ by 17. 5.

XVI. Converse ratio is when the antecedent is compared to the excess wherein the antecedent exceeds the consequent. *As* $A. B :: C. D.$ therefore by converse ratio. $A. A - B :: C. C - D.$ by the coroll. of the 19 of the 5.

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and also as many magnitudes in another order, comparing two to two being in the same ratio; it cometh to pass that as in the first order of magnitudes, the first is to the last, so in the second order of magnitudes is the first to the last. Or otherwise: it is comparison of the extremes together, the mean magnitudes being taken away.

XVIII. Ordinate proportionality is, when, as the antecedent is to the consequent, so is the antecedent to the consequent, and as the consequent is to any other, so is the consequent to any other. *As* $A. B :: D. E.$ also $B. C :: E. F.$ it shall be true also $A. C :: D. F$ by the 22. of the 5.

XIX. Inordinate proportion is, when three magnitudes being put, and others also, which are equal to these in multitude, as in the first magnitudes the antecedent is to the consequent, so in the second magnitudes is the antecedent to the consequent: and as in the first magnitudes the consequent is to any other, so in the second magnitudes any other thing to the antecedent. *As* $A. B :: F. G.$ also $B. C :: E. F.$ it shall be true in inordinate proportion. $A. C :: E. G.$ by the 23. of the 5.

XX. Any number of magnitudes being put; the proportion of the first to the last is compounded out of the proportions of the first to the second, the

the second to the third, and the third to the fourth, and so forwards till the proportion arise.

Let there be any number of magnitudes A, B, C, D. by this definition $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}$.

Axiom.

Magnitudes equimultiples to the same multiplex, are also equimultiples betwixt themselves.

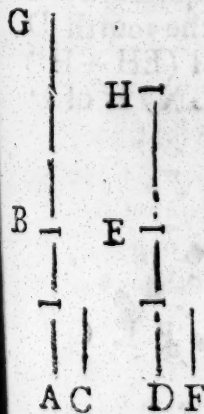
P R O P. I.



If there be a number of magnitudes how many soever, AB, CD equimultiples to a like number of magnitudes E, F each to other; how multiplex one magnitude AB is to one E, so multiples are all the magnitudes AB-CD to all the other magnitudes E+F.

Let AG, GH, HB the parts of the quantity AB, be equal to E, and also let CI, IK, KD the parts of the quantity CD be equal to F. The Number of these are put equal to those. Now whereas $AG + CI = E + F$; a and $GH + IK = E + F$; a and $HB + KD = E + F$ it is evident that $AB + CD$ doth so often contain $E + F$ as one AB contains E. Which was to be done.

P R O P. II.



If the first AB be equimultiplex to the second C, as the third DE is to the fourth F, and if the fifth BG be equimultiplex to the second C as the sixth EH is to the fourth F; then shall the first compounded with the fifth (AG) be equimultiplex to the second C, as the third compounded with the sixth (DH) is to the fourth F.

The number of parts in AB equal each to C is put equal to the numbers of part in DE, whereof each part

part is equal to F. Likewise the number of parts BG is put equal to the number of parts in EH. Therefore the number of parts in AB+BG is equal to the number of parts in DE+EH. *a* That is, the whole line AG is as equimultiplex of C, as the whole line DH is of F. Which was to be Demonstrated.

P R O P. III.

If the first A be equimultiplex of the second B, and the third C of the fourth D, and there be taken EI, EM equimultiples of the first and third, then will each of the magnitudes taken be alike equimultiplex of both, the one EI to the second B, the other FM to the fourth D.

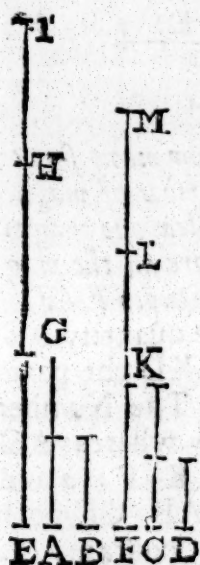
Let EG, GH, HI the parts of the multiplex EI be equal to A, also let FK, KL, LM the parts of the multiplex FM be equal to F, & the number of those is equal to the number of those. Moreover A (that is) EG or GH or HI is put as equimultiplex of B, as C, or FK, &c. of D. *b* Therefore EG+GH is equimultiplex

of the second B, as FK+KL is of the fourth D. *c* By the same way of argument is EI (EH+HI) as multiplex of B, as FM (FL+LN) is of D. Which was to be done.

a hyp.

b 2. 5.

c 2. 5.



P R O P.

P R O P. IV.

If the first A have the same ratio to the second B, as the third C to the fourth D; then also E and F the equimultiples of the first A and the third C, shall have the same ratio to G and H the equimultiples of the second B and the fourth D, according to any multiplication, if so taken as they answer each to other (E. G :: F. H.)

Take I and K the equimultiples of E and F; and also L and M the equimultiples of G and H. *a* Then is I as multiplex of A, as K of C; *a* and also L is as multiplex of B, as M of D. Therefore whereas it is A. B *b* :: C. D; according to the sixth definition, if I be $\square$, $\equiv$, $\supset$ L, then consequently after the same manner is K $\square$, $\equiv$, $\supset$ M, Therefore when I and K are taken as multiples of E and F, as L and M of G, and H, then will it be by the seventh definition E. G :: F. H. Which was to be demonstrated.

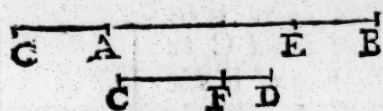


Coroll.

From hence is wont to be demonstrated the proof of inverse ratio.

For because A. B :: C. D, therefore if E $\square$, $\equiv$, $\supset$ G, then is likewise F $\square$, $\equiv$, $\supset$ H. therefore *c* 2. def. § it is evident that if G $\square$, $\equiv$, $\supset$ E, then is H $\square$, $\equiv$, $\supset$ F; *d* therefore B. A :: D. C. Which was to *d* 6. def. § be demonstrated.

P R O P. V.



If a magnitude AB be as multiplex of a magnitude CD, as a part taken from the one AE of a part taken from the other CF; the residue of the one EB shall be as multiplex of the residue of the other FD as the whole AB is of the whole CD.

Take the other GA, which shall be as multiplex to FD the residue, as AB is of the whole CD, or as the part taken away AE is of the part taken away CF. *a* Therefore the whole GA + AE is as multiplex of the whole CF + FD, as the one AE is of the one CF; that is, as AB of CD. therefore GE *b* = AB; and *c* so AE that was common being taken away, there remains GA = EB.

a 1. 5.

b 6. ax.

c 3. ax.

P R O P. VI.



If two magnitudes AB, CD be equimultiples of two magnitudes E, F; and some magnitudes AG and CH equimultiples of the same E, F, be taken away; then the residues GB, HD are either equal to these magnitudes E, F, or else equimultiples of them.

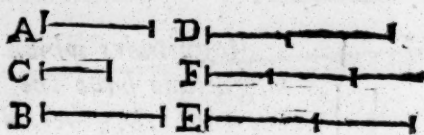
a 3. ax.

For because the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F; and also the number of parts in AG equal to the number of parts in CH; If from one you take AG, and from the other CH, *a* then remains the number of parts in the remainder GB equal to the number of parts in HD. therefore if GB be once E, then is HD once C. if GB be many times E, then is HD so of C. Which was to be demonstrated.

P R O P

PROP. VII.

Equal magnitudes A and B have to the same magnitude C the same proportion or ratio. And one and the same magnitude C hath the same ratio to equal magnitudes A and B.



Take D and E equimultiples of the equal magnitudes A and B, and F any-wise multiplex of C; then is $D = E$. Wherefore if $D < F$, a 6. ax. then also E will be $< F$. b 6. def. 5. $C :: B. C$ and c by inversion $C. A :: c C. B$. Which c cor. 4. 5. was to be demonstrated.

Schol.

If instead of the multiplex F, two equimultiples be taken, it shall be the same way proved that equal magnitudes have the same ratio to other magnitudes that are equal between themselves.

PROP. VIII.

Of unequal magnitudes AB, AC, the greater AB hath a greater ratio to the same third line D, than the lesser AC; and the same third line D hath a greater ratio to the lesser AC, than to the greater AB.

Take EF, EG equimultiples of the said AB, AC, so that EH being multiplex of D be greater than EG, but lesser than EF. (which will easily happen, if both EG and GF be taken greater than D.) It is manifest from 8 def. 5. that $\frac{AB}{D} < \frac{AC}{D}$ and $\frac{D}{AB} > \frac{D}{AC}$. Which was to be demonstrated.



P R O P. IX.



Magnitudes which to one and the same magnitude have the same ratio, are equal the one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equal one to the other.

a 8. 5.

1. Hyp. If $A.C :: B.C$; I say that $A=B$. For let A be greater or less than C. a then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. Which is contrary to the Hypothesis.

b 8. 5.

2. Hyp. If $C.B :: C.A$. I say that $A=B$. For let A be $\sqsubset$ B, b then $\frac{C}{B} \sqsubset \frac{C}{A}$. Which is against the Hypothesis.

P R O P. X.



Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude: and that magnitude to which the same carries a greater ratio, is the lesser magnitude.

a 7. 5.

b 8. 5.

1. Hyp. If $\frac{A}{C} \sqsubset \frac{B}{C}$. I say that $A \sqsubset B$. For if it be said that $A=B$, a then $A.C :: B.C$. which is contrary to the Hypothesis. If $A \sqsupset B$, b then is $\frac{A}{C} \sqsupset \frac{B}{C}$. Which is also against the Hypothesis.

c 7. 5.

d 8. 5.

2. Hyp. If $\frac{C}{B} \sqsubset \frac{C}{A}$ I say that $B \sqsupset A$. for if you say $B=A$, it's against the Hypothesis, for it will follow that $C.B :: C.A$. If you say $B \sqsubset A$, d then is $\frac{C}{B} \sqsupset \frac{C}{A}$. Which is also against the Hypothesis.

P R O P.

P R O P. XI.

G ———	H ———	I ———
A ———	C ———	E ———
B ———	D ———	F ———
K ———	L ———	M ———

Proportions which are one and the same to any third are also the same one to another.

Let $A : B :: E : F$, and $C : D :: E : F$. I say that $A : B :: C : D$. Take G, H, I , the equimultiples of A, C, E ; and K, L, M , the equimultiples of B, D, F . Now *a* because $A : B :: E : F$; if $G \square, =$, *a* *hyp.*
 $\neg K$. *b* then after the same manner $I \square, =$, $\neg M$.
 And likewise *a* because $E : F :: C : D$. if $I \square, =$,
 $\neg M$, *b* then is H likewise $\square, =$, $\neg L$. *c* where. *b* 6. def. 5.
 fore $A : B :: C : D$. Which was to be demonstrated. *c* 6. def. 5.

Schol.

Proportions that are one and the same to the same proportions, are the same betwixt themselves.

P R O P. XII.

G ———	H ———	I ———
A ———	C ———	E ———
B ———	D ———	F ———
K ———	L ———	M ———

If any number of magnitudes A, B; C, D; E and F be proportional; as one of the antecedents A is to one of the consequents B, so are all the antecedents A, C, E to all the consequents B, D, F.

Take the equimultiples of the antecedents G, H, I , and of the consequents K, L, M . Because that as multiplex as one G is of one A , *a* so multiplices *a* 1. 5.
 are all G, H, I , of all A, C, E ; and likewise as multiplex as one K is of one B , so multiplices are all K, L, M , of all B, D, F . Moreover because $A : B :: C : D$ *b* *hyp.*
 b *hyp.* if G be $\square, =$, or $\neg K$, then *b* *hyp.*
 will H likewise be $\square, =$, $\neg L$, and $I \square, =$, $\neg M$.
 and so if $G \square, =$, $\neg K$. in like manner will $G + H + I$ be $\square, =$, $\neg K + L + M$. *c* wherefore $A : B :: C : D$. *c* 6. def. 5.
 $A + C + E : B + D + F$. Which was to be dem.

G 3

Co.

Coroll.

From hence, if like proportionals be added to like proportionals, the wholes shall be proportional.

P R O P. XIII.

G_____	H_____	I_____
A_____	C_____	E_____
B_____	D_____	F_____
K_____	L_____	M_____

If the first A have the same ratio to the second B, that the third C hath to the fourth D, and if the third C have a greater proportion to the fourth D, than the fifth E to the sixth F; then also shall the first A have a greater proportion to the second B, than the fifth E to the sixth F.

Take G, H, I equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now because a 6. def. 5. that $A : B :: C : D$. if $H \sqsubset L$, a then is $G \sqsubset K$. but

b 8. def. 5. because $\frac{C}{D} \sqsubset \frac{E}{F}$, b it may be that $H \sqsubset L$, and yet

c 8. def. 5. I not $\sqsubset M$. c Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$. Which was to be demonstrated.

Schol.

But if $\frac{C}{D} \supset \frac{E}{F}$, then also is $\frac{A}{B} \supset \frac{E}{F}$. Also, if

$\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if

$\frac{A}{B} \supset \frac{C}{D} \supset \frac{E}{F}$, then is $\frac{A}{B} \supset \frac{E}{F}$.

P R O P.

PROP. XIV.

If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the first A be greater than the third C; then shall the second B be greater than the fourth D. But if the first A be equal to the third C, then the second B shall be equal to the fourth D. but if A be less, then is B also less.



a 8. 5.
b hyp.
c 13. 5.

Let $A \sqsubset C$. a then $\frac{A}{B} \sqsubset \frac{C}{B}$. b but $\frac{A}{B} = \frac{C}{D}$; c therefore $\frac{C}{B} \sqsubset \frac{C}{D}$. c there-
fore $B \sqsubset D$. By the like way of argument, if $A \supset C$, d then is $B \supset D$. But if A be put equal to C, then $C : B :: e A : B$ f $:: C : D$. g therefore $B = D$. Which was to be demonstrated.

d 10. 5.
e 7. 5.
f hyp.
g 11. 5. &
9. 5.

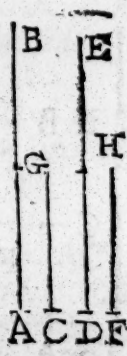
Schol.

By an argument a fortiori, if $\frac{A}{B} \supset \frac{C}{D}$, and $A \sqsubset C$, then is $B \sqsubset D$. Likewise if $A = B$, then is $C = D$. and if $A \sqsubset$, or $\supset B$, then also is $C \sqsubset$ or $\supset D$.

PROP. XV.

Parts C and F are in the same ratio, with their like multiples AB and DE, if taken correspondently. (AB. DE :: C. F.)

Let AG, GB parts of the multiplex AB be equal to C; and let DH, HE parts of the multiplex DE be equal to F. a The number of these parts is equal to the number of those. Therefore whereas b AG. C :: DH. F, and GB. C :: HE. F, therefore is c AG+GB (AB.) DH+HE (DE) :: C. F. Which was to be dem.

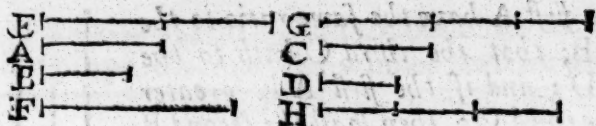


a hyp.
b 7. 5.
c 12. 5.

G 4

PROP.

P R O P. XVI.



If four magnitudes A, B, C, D be proportional, they also shall be alternately proportional (A. C :: B. D.)

Take E and F equimultiples of A and B; take also G and H equimultiples of C and D. Therefore E. F $a :: A. B$ $b :: C. D$ $a :: G. H$. Wherefore if E $\square, =, \triangleright G$, c then likewise is F $\square, =, \triangleright H$. Therefore A. C :: B. D. Which was to be dem.

a 15. 5.

b hyp.

c 11. 5. &

14. 5.

d 6. def. 5.

Schol.

Alternate ratio has place only then when the quantities are of the same kind. For heterogeneous quantities are not compared together.

P R O P. XVII.



a 1. 5.

b constr.

c 1. 5.

d 2. 5.

If magnitudes compounded be proportional (AB. CB :: DF. FE,) they shall be proportional also when divided. (AC. CB :: DE. FE.)

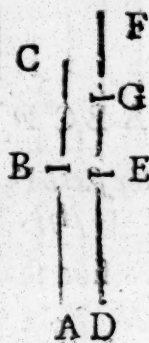
Take GH, HL, IK, KM, in order, the equimultiples of AC, CB, DF, FE; and also LN, MO, the equimultiples of CB, FE. The whole GL is a as multiplex of the whole AB, as one GH of one AC, b that is, as IK of DF, c or as the whole IM of the whole DE. Also HN (HL + LN) is d multiplex of CB, as KO (KM + MO) is of FE. Therefore, whereas by Hyp. AB. BC :: DE. EF. if GL be $\square, =, \triangleright$ HN, then like.

likewise e will $IM \sqsubset, =, \supset KO$. Take from e 6. def. 5, these HL, KM that are equal; and if the remainder GH be $\sqsubset, =, \supset LN$, f then will $IK \sqsubset, =, \supset f$ 5. ax. MO . g whence $AC.CB :: DF.FE$. Which was to g 6. def. 5, be demonstrated.

PROP. XVIII.

If magnitudes divided be proportional ($AB.BC :: DE.EF$.) the same also being compounded shall be proportional ($AC.CB :: DF.FE$.)

For if it can be, let $AB.CB :: DF$. B $FG \supset FE$. a Then by division will $AB.BC :: DG.GF$. b that is, $DG.GF :: DE.EF$. and being $DG \sqsubset DE$. c therefore is $GF \sqsubset EF$. Which is absurd. The like absurdity will follow if it be said $AB.CB :: DE.GF \sqsubset FE$.



a 17. 5.
 b hyp. $\&$
 i 5.
 c 14. 5.
 d 9. ax.

PROP. XIX.

If the whole AB be to the whole DE as the part AC taken away AC is to the part taken away DF , D then shall the residue CB be to the residue FE as the whole AB is to the whole DE .

Because a $AB.DE :: AC.DF$, b therefore by a hyp. permutation $AB.AC :: DE.DF$. c and thence by b 10. 5. division $AC.CB :: DF.FE$. b wherefore again by c 17. 5. permutation $AC.DF :: CB.FE$. d that is, $AB.DE :: CB.FE$. d hyp. $\&$ 11. 5. Which was to be demonstrated.

Coroll.

Hence, If like proportionals be subtracted from like proportionals, the residues shall be proportional.

2. Hence is converse ratio demonstrated.

Let $AB.CB :: DE.FE$. I say that $AB.AC :: DE.DF$. For by a permutation $AB.DE :: CB.FE$, b therefore $AB.DE :: AC.DF$, whence a gain by permutation $AB.AC :: DE.DF$. Which was to be demonstrated.

PROP.

The fifth Book of

P R O P. XX.



A B C D E F

If there be three magnitudes A, B, C, and others D, E, F equal to those in number, which being taken two and two in each order are in the same ratio, (A. B :: D. E; and B. C :: E. F,) and if of equality the first A be greater than the third C; then shall the fourth D be greater than the sixth F. But if the first A be equal to the third C, then the fourth D is so to the sixth F; and if A be less than C, so D is less than F.

a hyp.

b cor. 4. 5.

c hyp.

8. 5.

d schol. 13.

5.

e 10. 5.

1. Hyp. Let $A = C$. Because a $E F :: B. C$. by b inversion shall be $F. E :: C. B$. c But $\frac{C}{B} = \frac{A}{B}$, d there-

fore $\frac{F}{E} = \frac{A}{B}$ or $\frac{D}{E}$, e therefore $D = F$. Which was to be demonstrated.

2. Hyp. By the same way of argument, if $A > C$, it will appear that $D > F$.

f 7. 5.

g 11. 5.

9. 5.

3. Hyp. If $A = C$. Because $F. E :: C. B :: f$ $A. B :: D. E$. g therefore is $D = F$. Which was to be dem.

P R O P. XXI.



A B C D E F

If there be three magnitudes A, B, C, and others also D, E, F equal to them in number, which taken two and two are in the same ratio; and their proportion inordinate (A. B :: E. F. and B. C :: D. E.) and if of equality the first A be greater than the third C, then is the fourth D greater than the sixth F; but if the first be equal to the third, then is the fourth equal to the sixth; if less, so is the other likewise.

a hyp.

b 8. 5.

c schol. 13.

5.

d 10. 5.

1. Hyp. If $A < C$; then because a $D. E :: B. C$, therefore inversely $E. D :: C. B$. but $\frac{C}{B} < \frac{A}{B}$;

e therefore $\frac{E}{D} < \frac{A}{B}$ that is, $\frac{E}{F}$, d therefore $D < F$.

2. Hyp.

2. *Hyp.* By the like argument, if $A \supset C$, then is $D \supset F$.

3. *Hyp.* If $A = C$; then because $E. D :: e C. e 7. 5.$
 $B :: e A. B :: f E. F, g$ therefore is $D = F$. Which *f hyp.*
was to be demonstrated. g 9. 5.

P R O P. XXII.

If there be any number of magnitudes A, B, C, and others equal to them in number D, E, F, which taken two and two are in the same ratio ($A. B :: D. E.$ and $B. C :: E. F.$) they shall be in the same ratio also by equality, ($A. C :: D. F.$)

Take G, H equimultiples of A, D; and I, K of B, E; and also L, M of E, F.

Because *a* $A. B :: D. E.$
b therefore $G. I :: H. K.$
 and in like manner $I. L :: K. M.$ therefore, if $G \sqsubset, =, \supset, L, c$ then is $H \sqsubset, =, \supset M.$ *d* therefore $A. C :: D. F.$ By the same way of demonstration if further $C. N :: F. O$, then by equality $A. N :: D. O.$ Which was to be demonstrated.



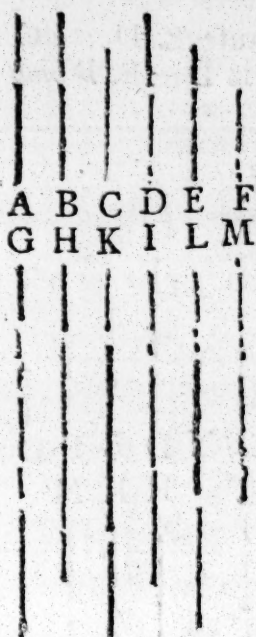
a hyp.
b 4. 5.

c 20. 5.
d 6. def. 5.

P R O P.

The fifth Book of

P R O P. XXIII.



a 15. 5.
b hyp.
c 4. 5.

If there be three magnitudes A, B, C, and others D, E, F, equal to them in number, which taken two and two are in the same ratio, and their proportionality inordinate (A. B :: E. F. and B. C :: D. E.) they shall be in the same ratio also by equality.

Take G, H, I equimultiples of A, B, D; and also K, L, M equimultiples of C, E, F. Then G. H :: a A. B :: b E. F. a :: L. M. Moreover because b B. C :: D. E, thence is c H. K :: I. L; therefore G, H, K, and I, L, M are according to 21. 5. Wherefore if G be $\square$, =, $\sqsupset$ K, then is likewise I

d 6. def. 5. $\square$, =, $\sqsupset$ M. and so d consequently A. C :: D. F. Which was to be demonstrated.

If there be more magnitudes than three, this way of demonstration holds good in them also.

Coroll.

*22. & 23. From hence * it follows that ratio's compounded of the same ratio's, are among themselves the same; as also that the same parts of the same ratio's, are among themselves the same.

P R O P. XXIV.

			If the first magnitude AB have the same ratio to the second C, which the third DE hath to the fourth F; and if the fifth BG have the same ratio to the second C, which the sixth EH hath to the fourth F, then shall the first compounded with the fifth (AG) have the same ratio to the second C, which the third compounded with the sixth (DH) hath to the fourth F.
A	B	G	
C	E	H	
D	F	F	

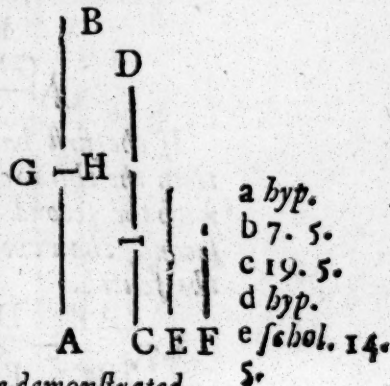
For

For because a $AB.C :: DE.F$, and by the Hyp. a *hyp.*
and inversion $C.BG :: F.EH$; therefore by b e - b 22. 5
quality $AB.BG :: DE.EH$. whence by com-
pounding, $AG.BG :: DH.EH$. Also c $BG.C :: c$ *hyp.*
 $EH.F$. Therefore again by b equality $AG.C ::$
 $DH.F$. Which was to be demonstrated.

P R O P. XXV.

If four magnitudes be proportional
($AB.CD :: E.F$) the greatest AB
and the least F shall be greater than the
rest CD , and E .

Make $AG = E$, and $CH = F$. Be-
cause $AB.CD :: a$ $E.F :: b$ $AG.CH$.
 c thence is $AB.CD :: GB.HD$.
 d but $AB = CD$. e therefore $GB =$
 HD . But $AG + F = E + CH$, therefore
 $AG + F + GB = E + CH + HD$, that
is, $AB + F = E + CD$. Which was to be demonstrated.



These propositions which follow are not Euclide's,
but taken out of other Authors, and here subjoyned
because of their frequent use.

P R O P. XXVI.

If the first have a greater A ——— C ———
proportion to the second, B ——— D ———
than the third to the fourth; E ———
then contrarywise, by con-
version, the second shall have a less proportion to the
first, than the fourth to the third.

Let $\frac{A}{B} > \frac{C}{D}$, I say that $\frac{B}{A} < \frac{D}{C}$. For conceive
 $\frac{C}{D} = \frac{E}{B}$, a therefore $\frac{A}{B} < \frac{E}{B}$. b whence $A < E$. c there-
fore $\frac{B}{A} > \frac{B}{E}$, d or $\frac{D}{C} < \frac{D}{E}$. Which was to be demonstrated.

a 13. 5.
 b 10. 5.
 c 8. 5.
 d cor. 4. 5.

P R O P. XXVII.

If the first have a greater pro- A ——— C ———
portion to the second, than the B ——— D ———
third to the fourth; then alter- E ———
nately the first shall have a greater
proportion to the third, than the second to the fourth.

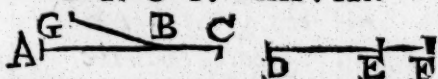
Let

The fifth Book of

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. then I say $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceive

a 10. 5. $\frac{E}{B} = \frac{C}{D}$. a therefore $A \sqsubset E$, b and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$
 b 1. 5. $\frac{B}{D}$
 c 16. 5. c or $\frac{B}{D}$. Which was to be demonstrated.

P R O P. XXVIII.

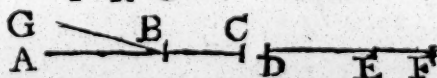


If the first have a greater proportion to the second than the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, than the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I say that $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10. 5. ceive $\frac{GB}{BC} = \frac{DE}{EF}$. a therefore is, $AB \sqsubset GB$. add BC
 b 4. ax. to each part, then b will $AC \sqsubset GC$. c therefore
 c 8. 5. $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$. d that is, $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. Which was to be dem.
 d 18. 5.

P R O P. XXIX.



If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, than the third to the fourth.

Let $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. then I say $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For con-

a 10. 5. ceive $\frac{GC}{BC} = \frac{DE}{EF}$. a therefore $AC \sqsubset GC$. Take a-
 b 5. ax. way BC, that is common; then b remains $AB \sqsubset$
 c 8. 4- GB . c Therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$. d or $\frac{DE}{EF}$. Which was to be
 d 17. 5. demonstrated.

P R O P.

PRO P. XXX.

If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth hath to the fourth, then by converse ratio shall the first compounded with the second have a lesser ratio to the first, than the third compounded with the fourth shall have to the third.

Let $\frac{AC}{BC} \supset \frac{DF}{EF}$. Then I say that $\frac{AC}{AB} \supset \frac{DF}{DE}$. For

because that $\frac{AC}{BC} \supset \frac{DF}{EF}$. *b* therefore by division *a* hyp. *b* 29. 5.

$\frac{AB}{BC} \supset \frac{DE}{EF}$. by conversion *c* therefore $\frac{BC}{AB} \supset \frac{EF}{DE}$ *c* 26. 5.

and *d* therefore by composition $\frac{AC}{AB} \supset \frac{DF}{DE}$. Which *d* 28. 5. was to be demonstrated.

PRO P. XXXI.

If there be three magnitudes A, B, C, and others also D, E, F, equal to them in number; and if there be a greater proportion of the first of the former to the second, than there is of the first of the last to their second

$\left(\frac{A}{B} \supset \frac{D}{E} \right)$; and there be also a greater proportion of the second of the first magnitudes to the third, than there is of the second of the last magnitudes to their third

$\left(\frac{B}{C} \supset \frac{E}{F} \right)$. Then by equality also shall the ratio of the first of the former magnitudes to the third be greater than the ratio of the first of the latter magnitudes to the third

$\left(\frac{A}{C} \supset \frac{D}{F} \right)$

Co.

a 10. 5. Conceive $\frac{G}{C} = \frac{E}{F}$. a therefore is $B \sqsubset G$, and b
b 8. 5.

c 13. 5. therefore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{F}$. c there-

d 10. 5. fore $\frac{H}{G} \supset \frac{A}{B}$. therefore much more $\frac{H}{G} \supset \frac{A}{G}$. d where-

e 8. 5. fore $A \sqsubset H$, e and consequently $\frac{A}{C} \sqsubset \frac{H}{C}$. f or $\frac{D}{F}$.

f 12. 5.

P R O P. XXXII.

If there be three A ——— D ———
magnitudes A, B, C, B ——— E ———
and others D, E, F, e C ——— F ———
equal to them in num- G ———
ber; and there be a H ———
greater proportion of the first of the former mag-
nitudes to the second, than there is of the se-
cond of the latter to the third $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ and also
the ratio of the second of the former to the third be
greater than the ratio of the first of the latter to the
second $\left(\frac{B}{C} \sqsubset \frac{D}{E}\right)$ then by equality also shall the
proportion of the first of the former to the third, be
greater than that of the first of the latter to the third
 $\left(\frac{A}{C} \sqsubset \frac{D}{E}\right)$

The demonstration of this Proposition is alto-
gether like that of the precedent.

P R O P. XXXIII.

E
A ——— I ——— B If the proportion of the whole AB
C ——— I ——— D to the whole CD be greater than the
F proportion of the part taken away
AE to the part taken away CF;
then shall also the ratio of the remain-
der EB to the remainder FD be greater than that of the
whole AB to the whole CD.

Because

Because that $\frac{AB}{CD} a \sqsubset \frac{AE}{EF} b$ therefore by permuta *a hyp.*
 $\frac{AB}{CD} \sqsubset \frac{AE}{EF}$ *b 27. 5.*

tation $\frac{AB}{AE} \sqsubset \frac{CD}{CF} c$ therefore by converse ratio *c 30. 5.*

$\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$.

Which was to be demonstrated.

P R O P. XXXVI.

If there be any $A \text{ --- } D$
 number of magnitudes, $B \text{ --- } E$
 and others also equal $C \text{ --- } F$
 to them in number; $G \text{ --- } H$
 and the proportion of
 the first of the former to the first of the latter be
 greater than that of the second to the second, and that
 greater than the proportion of the third to the third,
 and so forward: all the former magnitudes together
 shall have a greater ratio to all the latter together,
 than all the former, leaving out the first, shall have
 to the latter, leaving out the first; but less than that
 of the first of the former to the first of the latter;
 and lastly, greater than that of the last of the former
 to the last of the latter.

You may please to consult Interpreters for the
 demonstration hereof, we having for brevity sake
 omitted it, and because 'tis of no use in these
 Elements.

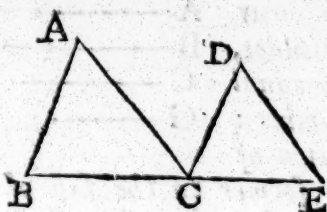
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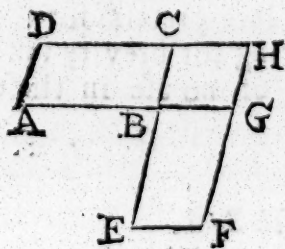
The SIXTH BOOK
OF
E U C L I D E ' S
E L E M E N T S.

Definitions.



I. **L**ike right-lined figures (ABC, DCE) are such whose several angles are equal one to the other, and also their sides about the equal angles, proportional.

The angle B = DCE, and AB. BC :: DC. CE. Also the angle A = D, and BA. AC :: CD. DE. Lastly the angle ACB = E, and BC. CA :: CE. ED.

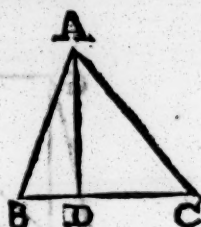


II. Reciprocal figures are (BD, BF) when in either figure are the terms antecedents and consequents of ratio's (that is, AB. BG :: EB. BC.)

III. A right line AB is said to be cut according to extreme and mean proportion, when as the whole AB is to the greater segment AC, so is the greater segment AC to the less CB (AB. AC :: AC. CB.)



IV. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the base BC.

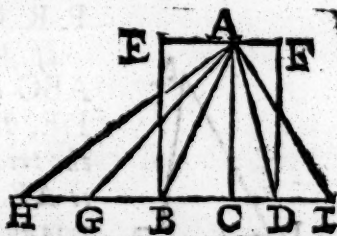


V. A ratio is said to be compounded of two ratio's, when the quantities of the ratio's, being multiplied the one into the other, do produce any ratio. As the ratio of A to C is compounded of the ratio's

of A to B and B to C. For $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$ *a 20. def. 5.*
 $\frac{A}{B} = \frac{AB}{BC}$ *b 15. 4.*

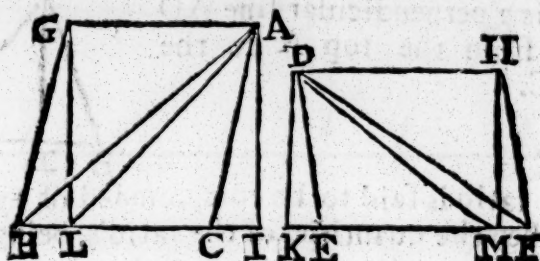
P R O P. I.

Triangles ABC, ACD, and parallelograms BCAE, CDEA, which have the same height, are in proportion one to the other, as their bases, BC, CD are.



a Take as many as you please, BG, GH, equal *a 3. 1.* to BC, and also DI = CD. and join AG, AH, AI.

b The triangles ACB, ABG, AGH are equal, *b 38. 1.* and *b* also the triangle ACD = ADI. Therefore the triangle ACH is as multiplex of the triangle ACB, as the base HC is of the base BC; and the triangle ACI as multiplex of the triangle ACD, as the base CI is of CD. But if HC =, =, CI, *c* then is likewise the triangle AHC =, =, *c sch. 38. 1.* *d* ACI; and *d* therefore BC. CD :: the triangle *d 6. def. 5.* ABC. ACD :: *e* Pgr. CE. CF. Which was to be *e 41. 1. 5.* demonstrated.



Hence, *Triangles ABC, DEF, and Pgrs. AGBC, DEFH, whose bases BC, EF are equal, are in such proportion as their altitudes, AI, DK are.*

a 3. 1.

b 7. 5.

c 1. 6.

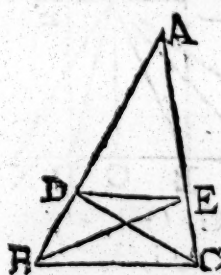
d 41. 1. &

15. 5.

a Take $IL = CB$, and $KM = EF$; and join LA, LG, MD, MH. then is it evident that the triangles $ABC.DEF :: b ALI.DKM :: c AI.DK :: d$ pgr. $AGBC.DEFH$. Which was to be demon.

PROP. II.

If to one side BC of a triangle ABC be drawn a parallel right line DE, the same shall cut the sides of the triangle proportionally ($AD.BD :: AE.EC$.) And if the sides of the triangle be proportionally cut ($AD.BD :: AE.EC$.) then a right line DE joined at the sections D, E, shall be parallel to the remaining side of the triangle BC. Draw CD and BE.



a 37. 1.

b 7. 5.

c 2. 6.

d 11. 5.

e 1. 6.

f 9. 5.

g 39. 1.

1. *Hyp.* Because the triangle $DEB = DEC$, b therefore shall be the triangle $ADE.DBE :: ADE.ECD$. But the triangle $ADE.DBE :: c AD.DB$, and the triangle $ADE.DEC :: AE.EC$, d therefore $AD.DB :: AE.EC$.

2. *Hyp.* Because $AD.DB :: AE.EC$. e that is the triangle $ADE.DBE :: ADE.ECD$; f therefore is the triangle $DBE = ECD$; and g therefore DE, BC are parallels. Which was to be demonstrated.

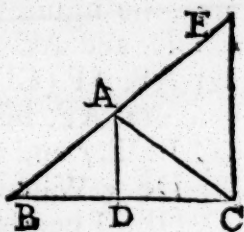
Schol.

If there be drawn many parallels to one side of any triangle, then all the segments of the sides shall

shall be proportional; as is easily deducible from the precedent.

P R O P. III.

If an angle BAC of a triangle BAC be bisected, and the right line AD, that bisects the angle, cut the base also; then shall the segments of the base have the same ratio that the other sides of the triangle have, (BD. DC : AB.



AC.) And if the segments of the base have the same ratio, that the other sides of the triangle have (BD. DC :: AB. AC.) then a right line AD drawn from the top A to the section D, shall bisect that angle BAC of the triangle.

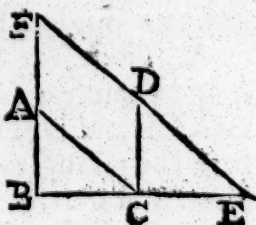
Produce BA, and make AE = AC, and join CE.

1. Hyp. Because AE = AC, therefore is the angle ACE a = E b = $\frac{1}{2}$ BAC c = DAC; d there- a 5. 1. fore DA, CE are parallels. e Wherefore BA. AE. b 32. 1. (AC) :: BD. DC. c hyp.

2. Hyp. Because BA. AC (AE) :: BD. DC, f d 27. 1. therefore are DA, CE parallels; and g therefore e 2. 6. is the angle BAD = E; and the angle DAC g = f 2. 6. ACE h = E. k therefore the angle BAD = DAC. g 29. 1. Wherefore the angle BAC is bisected. Which was h 5. 1. to be demonstrated. k 1. ax.

P R O P. IV.

Of equiangular triangles ABC, DCE, the sides are proportional which are about the equal angles, B, DCE (AB. BC : DC. CE, &c.) And the sides AB, DC, &c. which are subtended under the equal angles ACB, E, &c. are homologous, or of like ratio.



Set the side BC in a direct line to the side CE, and produce BA and ED till they meet.

H 3

Be-

2 32.

b *hyp.*

c 28. 1.

d 34. 1.

e 2. 6.

f 16. 5.

g 22. 5.

Because the angle $Bb = ECD$, c therefore BF , CD are parallel: Also because the angle $BCA b = CED$, e therefore are CA , EF parallel Therefore the figure $CAFD$ is a Pgr. d therefore $AF = CD$. and $AC = d FD$. Whence it is evident that $AB. AF (CD) :: e BC. CE$. f by permutation therefore $AB. BC :: CD. CE$. also $BC. CE :: FD (AC) DE$. f and thence by permutation $BC. AC :: CE. DE$. g Wherefore also by equality $AB. AC :: CD. DE$. Therefore, &c.

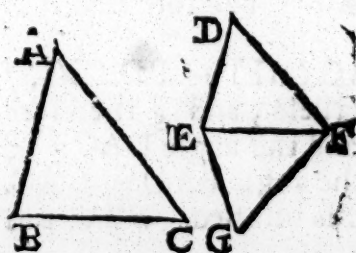
Coroll.

Hence $AB. DC :: BC. CE :: AC. DE$.

Schol.

Hence, If in a triangle FBE there be drawn AC a parallel to one side FE , the triangle ABC shall be like to the whole FBE .

P R O P. V.



If two triangles ABC , DEF have their sides proportional ($AB. BC :: DE. EF$, and $AC. BC :: DF. EF$, and also $AB. AC :: DE. DF$) those triangles are equiangular, and those angles equal under which

are subtended the homologous sides.

a 23. 1.

b 32. 1.

c 4. 6.

d *hyp.*

e 11. 5. and

9. 5.

i 8. 1.

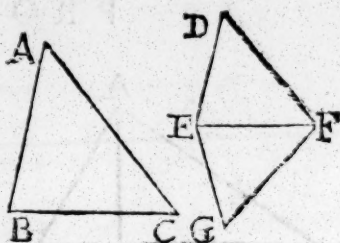
g 32. 1.

At the side EF make the angle $FEG = B$, and the angle $EFG = C$; b whence the angle $G = A$. Therefore $GE. EF c :: AB. BC :: d DE. EF$. e and therefore $GE = DE$. Likewise $GF. FE c :: AC. CB :: d DF. FE$. e therefore $GF = DF$. Therefore the triangles DEF, GEF are mutually equilateral. f Therefore the angle $D = G = A$, and the angle $FED f = FEG = B$, and g consequently the angle $DFE = C$. Therefore, &c.

P R O P.

PROP. VI.

If two triangles ABC, DEF have one angle B equal to one angle DEF, and the sides about the equal angles B, DEF proportional (AB. BC :: DE. EF) then those triangles ABC, DEF are equian-

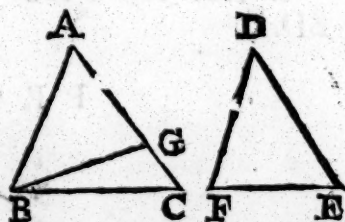


gular, and have those angles equal, under which are subtended the homologous sides.

At the side EF make the angle FEG = B, and the angle EFG = C; a then will the angle G = A. a 32. 1. Therefore GE. EF :: b AB. BC :: c DE. EF, d and b 4. 6. therefore DE = GE. But the angle DEF e = B f = c hyp. GEF; therefore the angle D g = G = A, h and d 9. 5. consequently the angle EFD = C. Which was to e hyp. be demonstrated. f constr.

PROP. VII.

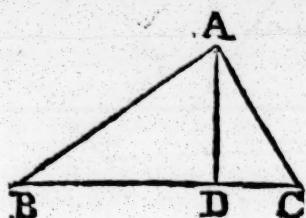
If two triangles ABC, DEF have one angle A equal to one angle D, and the sides about the other angles ABC, E, proportional (AB. BC :: DE. EF) and if they have both of the remaining angles C, F either less or not less than a right angle; then shall the triangles ABC, DEF be equiangular, and have those angles equal about which the proportional sides are.



For, if it can be, let the angle ABC = E, and make the angle ABG = E. Therefore, whereas the angle A a = D. b thence is the angle ACB = F. a hyp. Therefore AB. BG c :: DE. EF :: d AB. BC. e there- b 32. 1. fore BG = BC. f therefore the angle BGC = BCG. c 4. 6. g Therefore BGC or C is less than a right angle, d hyp. and h consequently AGB or F is greater than a e 9. 5. right: Therefore the angles C and F are not of f 5. 1. the same species or kind, which is against the g cor. 17. 1. Hypothesis. h cor. 13. 1.

The sixth Book of

P R O P. VIII.



If in a right angled triangle ABC, from the right angle BAC there be drawn AD a perpendicular to the base BC; then the triangles about the perpendicular (ADB, ADC) are like both

to the whole triangle ABC, and also one to the other.

a hyp.

b 12. ax.

c 32. and

d 6.

d 21. 6.

For because BAC, ADB are a right angles, b and so equal, and B common; the triangles BAC, ADB are like. By the same argument BAC, ADC are like; d whence also ADB, ADC will be like. Which was to be demonstrated.

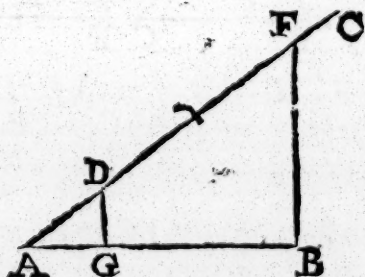
Coroll.

c 1. def. 6.

Hence, 1. BD. DA ϵ :: DA. DC.

2. BC. AC :: AC. DC. and CB. BA :: BA. BD.

P R O P. IX.



a 3. 1.

b 31. 1.

c 2. 6.

d 18. 5.

DE, FF. join FB, to which from D draw the parallel DG; and the thing is done.

For GB. AG :: FD. AD; whence by composition AB. AG :: AF. AD. therefore, whereas $AD = \frac{1}{3}$ of AF, therefore is $AG = \frac{1}{3}$ of AB. Which was to be done.

P R O P.

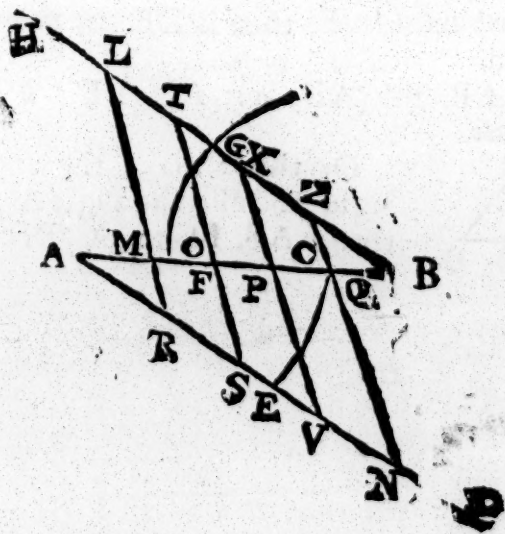
PROP. X.

To divide a right line given
AB not divided (in F and G)
as another line given AC was
cut (in D and E.)

Let a right line BC join the extremities of the line divided, and of the line not divided; and to that line from the points E, D draw the parallels EG, DF meeting with the right line that is to be cut, in G and F; Then the thing is done, a 31. 1.

For let DH be a drawn parallel to AB. Then AB
DE :: b AF. FG. and DE. EC b :: DI. IH : c FG. b 2. 6.
GB. Which was to be done. c 34. 1. E

Schol.



Hence is learnt to cut a right line given **AB** into as many equal parts as you please (suppose 5;) which will be more easily performed thus.

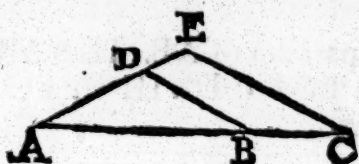
Draw an infinite line AD, and another BH parallel to it and infinite also. Of these take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL; in

in each line less parts by one, than are required in AB; then let the right lines LR, TS, XV, ZN, be drawn; these lines so drawn shall cut the right line given AB into five parts.

a 33. r.
b constr.
c 2. 6.

For RL, ST, VX, NZ are *a* parallels; therefore, whereas AR, RS, SV, VN are *b* equal; *c* thence AM, MO, OP, PQ are equal also. Likewise, because that BZ = ZX, therefore is BQ = PQ. and therefore AB is cut into five parts. Which was to be done.

P R O P. XI.



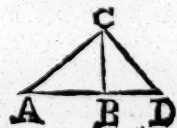
Two right lines being given AB, AD, to find out a third in proportion to them (DE.)

Join BD, and from AB being produced take BC = AD. Through C draw CE parallel to BD; with which let AD produced meet in E. then is DE the proportional required.

a 2. 6.

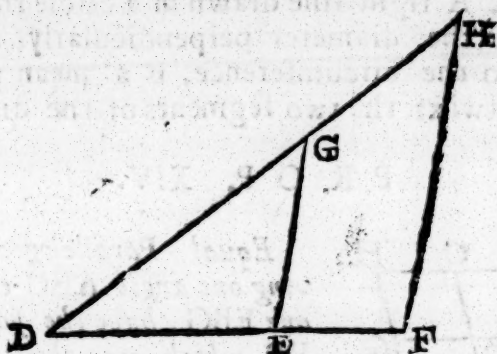
For AB. BC (AD) *a* :: AD. DE. Which was to be done.

b cor. 8.



Or thus: make the angle ABC right, and also the angle ACD right, then *b* AB. BC :: BC. BD.

PROP. XII.

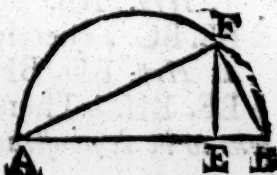


Three right lines being given DE, EF, DG, to find out a fourth proportional GH.

Join EG, and thro' F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that $DE : EF :: DG : GH$. Which a 2. 6. was to be done.

PROP. XIII.

Two right lines being given AE, EB, to find out a mean proportional EF.

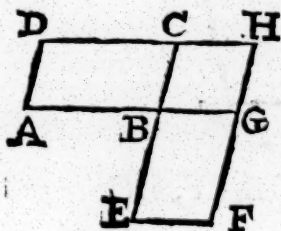


Upon the whole line AB as a diameter describe a semicircle AFB, and from E erect a perpendicular FF meeting with the periphery in F. then $AE : EF :: EF : EB$. For let AF and FB be drawn; a then from the right angle of the right-angled triangle AFB is drawn a right line FE perpendicular to the base. b Therefore $AE : FE :: FE : EB$. Which was to be done. a 31. 3. b cor. 8. 6.

Coroll.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportional betwixt the two segments of the diameter.

P R O P. XIV.



Equal Parallelograms having one angle ABC equal to one EBG, have the sides BD, BF which are about the equal angles reciprocal (AB. EG :: EB. BC;) and those parallelograms BD, BF which have one angle ABC equal to one EBG, and the sides which are about the equal angles reciprocal, are equal.

For let the sides AB, BG about the equal angles *a sch. 15. 1.* make one right line; a wherefore EB, BC shall do the same. Let FG, DC be produced till they meet.

b 1. 6.

c 7. 5.

d 1. 6.

e 11. 5.

f 1. 6.

g hyp.

h 1. 6.

k 11. and

9. 5.

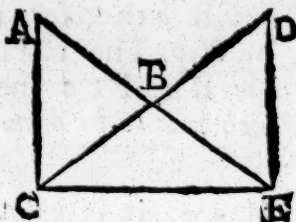
1. Hyp. AB. BG *b* :: BD. BH :: *c* BF. BH :: d BE. BC. *e* therefore, &c.

2. Hyp. BD. BH :: *f* AB. BG :: *g* BE. BC :: *h* BF. BH. *k* Therefore the Pgr. BD = BF. Which was to be demonstrated.

P R O P.

P R O P. XV.

Equal triangles having one angle $\angle ABC$ equal to one $\angle DBE$, their sides which are about the equal angles are reciprocal ($AB.BE :: DB.BC$.) And those triangles that have one angle $\angle ABC$ equal to one



$\angle DBE$, and have also the sides that are about the equal angles reciprocal ($AB.BE :: DB.BC$.) are equal.

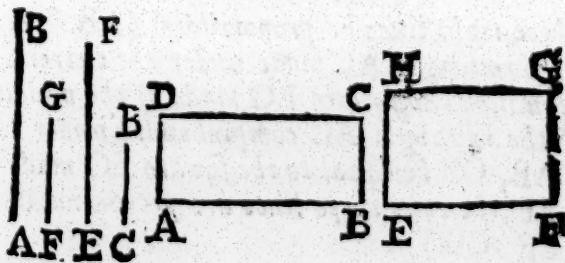
Let the sides CB, BC , which are about the equal angles be set in a straight line; a therefore ABE is a right line. Let CE be drawn.

1. Hyp. $AB.BE :: b$ the triangle $ABC.CBE$ $c ::$ the triangle $DBE.CBE :: d$ $DB.BC. e$ therefore, $\&c.$

2. Hyp. The triangle $ABC.CBE :: f$ $AB.BE :: g$ $DB.BC h ::$ the triangle $DBC.CBE. k$ Therefore the triangle $ABC = DBE$. Which was to be demonstrated.

a sch. 15. 1.
b 1. 6.
c 7. 5.
d 1. 6.
e 11. 5.
f 1. 6.
g hyp.
h 1. 6.
k 11. and
9. 5.

P R O P. XVI.



If four right lines be proportional ($AB.FG :: EF.CB$) the rectangle AC comprehended under the extremes AD, CB , is equal to the rectangle EG comprehended under the means FG, EF . And if the rectangle AC comprehended under the extremes AB, CB be equal to the rectangle EG comprehended under the means FG, EF , then are the four right lines proportional. ($AB.FG :: EF.CB$.)

1. Hyp.

a 12. ax.

1. Hyp. The angles B and F are right, and a consequently equal, and by hypothesis $AB \cdot FG :: EF \cdot CB$. b therefore the rectangle $AC = EG$.

b 14. 6.

c hyp.

d 14. 6.

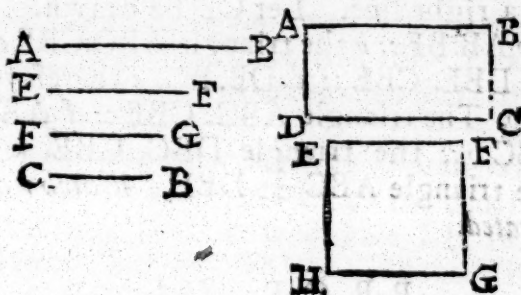
2. Hyp. The rectangle $AC c = EG$, and the angle $B = F$; d therefore $AB \cdot FG :: EF \cdot CB$, Which was to be demonstrated.

Coroll.

e 12. 6.

Hence, it is easy to apply a rectangle given EG to a right line given AB ; (*viz.*) e by making $AB \cdot EF :: FG \cdot BC$.

P R O P. XVII.



If three right lines be proportional ($AB \cdot EF :: EF \cdot CB$.) the rectangle AC made under the extremes AB , CB is equal to the square EG made of the middle EF . And if the rectangle AC comprehended under the extremes AB , CB be equal to the square EG made of the middle EF , then the three lines are proportional, ($AB \cdot EF :: EF \cdot CB$.)

Take $FG = EF$.

a hyp.

b 16. 6.

c 29. def. 1.

d hyp.

e 16. 6.

1. Hyp. $AB \cdot EF :: a \cdot EF$ (FG .) CB . therefore the rectangle $AC b = EG c = EFq$.

2. Hyp. The rectangle $AC d =$ to the square $EG = EFq$. e therefore $AB \cdot EF :: FG$ (EF .) BC .

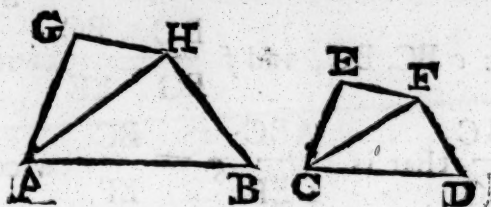
Which was to be demonstrated.

Coroll.

Let $A \times B = Cq$. therefore $A \cdot C :: C \cdot B$.

P R O P.

PROP. XVIII.



From a right line given AB to describe a right-lin'd figure AGHB like and alike situate to a right-lin'd figure given CEF D.

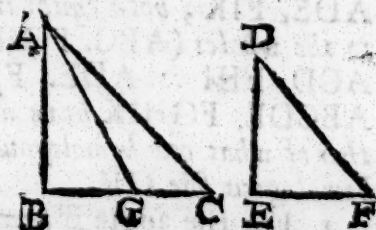
Relolve the right-lined figure given into triangles; *a* Make the angle $ABH = D$, *a* and the *a* 23. 1. angle $BAH = DCF$, *a* and the angle $AHG = CFE$, *a* and the angle $HAG = FCE$. then AGHB shall be the right-lined figure sought.

For the angle B *b* $= D$, and the angle BAH *b* *constr.* $b = DCF$. *c* wherefore the angle $AHB = CFD$. *c* 32. 1. *b* also the angle $HAG = FCE$, and the angle AHG *b* $= CFE$. *c* wherefore the angle $G = E$, and the whole angle GAB *d* $= ECD$, and the *d* 2. *ax.* whole angle GHB *d* $= EFD$. The Polygons therefore are mutually equiangular. Moreover because the triangles are equiangular, therefore AB . BH *e* $:: CD$. DF ; and AG . GH *e* $:: CE$. EF . Like- *e* 4. 6. wise AG . AH $:: CE$. CF . and AH AB $:: CF$. CD . *f* From whence by equality AG . AB $:: CF$. *f* 22. 5. CD . After the same manner GH . HB $:: EF$. FD . *g* 6. *def.* 5. Therefore the Polygons ABHG, CDFE are like and alike situate. Which was to be done.

PROP. XIX.

Like triangles ABC, DEF are in duplicate ratio of their homologous sides, BC, EF.

a Let there be made BC . $EF :: EF$. BG . and let AG . be drawn. Because that AB . DE *b* $:: BC$. EF *c* $:: b$ 4. 6.



EF. *c* *constr.*

d 15. 6. EF. BG, and the angle $B = E$. *d* therefore is the triangle $ABG = DEF$. But the triangle ABC,

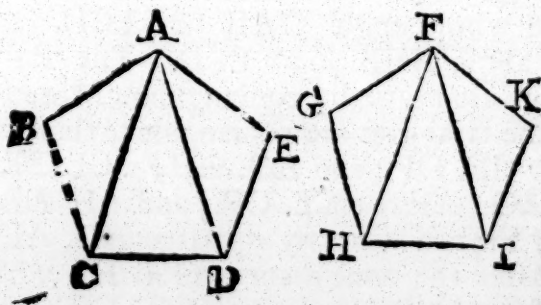
e 1. 6. $ABG :: e$ BC. BG, and *f* $\frac{BC}{BG} = \frac{BC}{EF}$ twice; there-
f 10. def. 5.

g 11. 5. fore $\frac{ABC}{ABG}$ that is $\frac{ABC}{DEF} g = \frac{BC}{EF}$ twice. Which
was to be demonstrated.

Coroll.

Hence, If three right lines (BC, EF, BG) be proportional, then as the first is to the third, so is a triangle made upon the first BC, to a triangle like and alike described upon the second EF; or so is a triangle described upon the second EF to a triangle like and alike described upon the third.

P R O P. XX.



Like Polygons ABCDE, FGHK are divided into equal triangles ABC, FGH, and ACD, FHI, and ADE, FIK; both equal in number and homologous to the wholes ($ABC. FGH :: ABCDE. FGHK :: ACD. FHI :: ADE. FIK$.) And the Polygons ABCDE, FGHK have a double ratio one to the other of what one homologous side BC hath to the other homologous side GH.

a hyp.
b 6. 6.

1. For the angle $B = G$, and $AB. BC :: FG. GH$. *b* therefore the triangles ABC, FGH are

are equiangular. After the same manner are the triangles AED, FKI like. Whereas therefore the angle BCA $b =$ GHF, and the angle ADE $b =$ FLK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are c equal, there remains the angle ACD $d =$ FHI, and the angle ADC $=$ FIH; e from whence also the angle CAD $=$ HFI, therefore the triangles ACD, FHI are like. Therefore, &c.

b 6. 6.

c hyp.

d 3. ax.

e 32. 1.

2. Because that the triangles BCA, GHF are like,

therefore is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the same reason is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; lastly $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice. now whereas that BC. GH $g ::$ CD. HI $g ::$ DE. IK. b therefore is the triangle BCA. GHF g h $sch. 23. 5.$ CAD. HFI h $sch. 23. 5.$ DEA. IKF k $12. 5.$:: the polygon ABCDEFGHIK $12. 5.$

fig. 6.

g hyp. and

16. 5.

h sch. 23. 5.

k 12. 5.

$\frac{BC}{GH}$ twice.

Coroll.

1. Hence, If there be three right lines proportional, then as the first is to the third, so is a polygon made upon the first to a polygon made on the second like and alike described; or so is a polygon described upon the second to a polygon made on the third like and alike described.

By which is found out a method of enlarging or diminishing any right lined figure in a ratio given: As if you would make a pentagon quintuple of that pentagon whereof CD is the side, then betwixt AB and 5 AB find out a mean proportional, * upon this raise a pentagon like to that given, and it shall be quintuple of the pentagon given.

* 18. 6.

2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, viz. by finding out a third proportional.

I

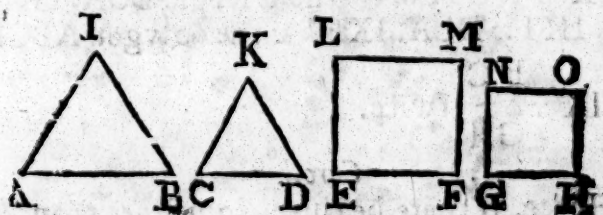
PROP.



Right lined figures ABC, DIE which are like to the same right-lined figure HFG, are also like one to the other.

For the angle A $a = H a = D$; and the angle C $a = G a = E$; and the angle B $a = F a = I$. Also $a AB.AC :: HF.HG :: DI.DE$ and $a AC.CB :: HG.GF :: DE.EI$. And $AB.BC :: HF.FG :: DI.IE$. Therefore $a ABC, DIE$ are like. Which was to be dem.

P R O P. XXII.



If four right lines be proportional ($AB.CD :: EF.GH$) the right-lined figures also described upon them being like and in like sort situate, shall be proportional ($ABI.CDK :: EM.GO$.) And if the right-lined figures described upon the lines, like and alike situate, be proportional ($ABI.CDK :: EM.GO$) then the right lines also shall be proportional ($AB.CD :: EF.GH$).

a 19. 6.

$$1. Hyp. \frac{ABI}{CDK} a = \frac{AB}{CD} \text{ twice} = a \frac{EF}{GH} \text{ twice} a = \frac{EM}{GO}$$

b hyp.

b therefore $ABI.CDK :: EM.GO$.

c 20. 6.

$$2. Hyp. \frac{AB}{CD} \text{ twice} a = \frac{ABI}{CDK} b = \frac{EM}{GO} = c \frac{EF}{GH}$$

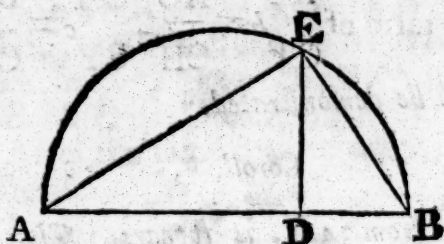
d cor. 23. 5.

twice. Therefore $AB.CD :: EF.GH$. Which was to be demonstrated. Schol.

Schol.

Hence is deduced the manner and reason of multiplying surd quantities. ex. g. Let $\sqrt{5}$ be to be multiplied into $\sqrt{3}$. I say that the product will be $\sqrt{15}$. For by the definition of multiplication it ought to be, as 1. $\sqrt{3} :: \sqrt{5}$. to the product Therefore by this q. 1. q. $\sqrt{3} :: q. \sqrt{5}$. q. of the product. That is 1. 3 :: 5. to the square of the product. Therefore the square of the product is 15. Wherefore $\sqrt{15}$ is the product of $\sqrt{3}$ into $\sqrt{5}$. Which was to be dem.

THEOREM.



If a right line AB be cut any-wise in D, the rect- angle comprehended under the parts AD, DB is a mean proportional betwixt their squares. Likewise the rect- angle comprehended under the whole AB and one part AD, or DB, is a mean proportional betwixt the square of the whole AB and the square of the said part AD, or DB. Pet. Herig.

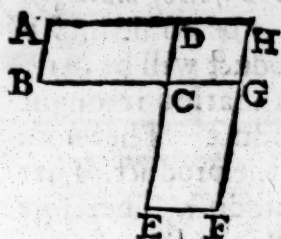
Upon the diameter AB describe a semicircle; from D erect a perpendicular DE meeting with the periphery in E. join AE, BE.

It's evident that AD. DE $a ::$ DE. DB. b therefore ADq. DEq $::$ DEq. DBq. c that is, ADq. ADB $::$ ADB. DBq. Which was to be demonstrated. a cor. 8. 6.
b 22. 6.
c 17. 6.

Moreover BA. AE $::$ AE. AD. e therefore BAq. AEq $::$ AEq. ADq. f that is, BAq. BAD $::$ BAD. ADq. After the same manner ABq. ABD $::$ ABD. BDq. Which was to be demonstrated. d cor. 8. 6.
e 22. 6.
f 17. 6.

Or thus: suppose $Z = A + E$. It is manifest that Aq. AE $::$ a A. E. $::$ a AE. Eq. also Zq. ZA $::$ a Z. A $::$ ZA. Aq. and Zq. ZE $::$ a Z. E $::$ ZE. Eq. a 1. 6.

P R O P. XXIII.



Equiangular parallelograms AC, CF, have the ratio one to the other, which is compounded of their sides.

$$\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CE} \right)$$

a sch. 15. Let the sides about the equal angles C be a set in a direct line, and let the Pgr. CH be completed.

b 20. def. 5. Then is the ratio of $\frac{AC}{CF} = \frac{AC}{CH} + \frac{CH}{CF} = \frac{BC}{CG} + \frac{DC}{CE}$

c 1. 6.

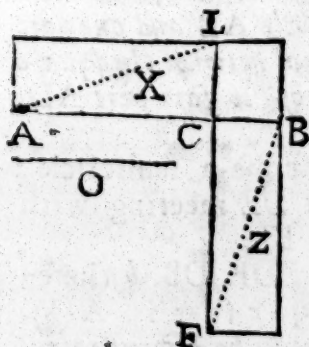
Which was to be demonstrated.

Coroll.

Andr.

Tacq. 15. 5. Hence, and from 34. 1. it appears, 1. That triangles which have one angle equal (as C) have a ratio compounded of the ratio's of the right lines, (AC to CB, and LC to CF,) containing the equal angle.

* 35. 1.



2. That all rectangles, and * consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude. After the like manner you may argue in triangles.

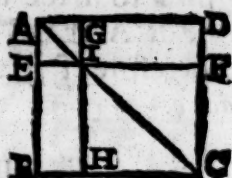
3. From hence is apparent how to give the proportion of

triangles and parallelograms. Let there be two Pgrs. X and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O. * then will it be X. Z :: AC. O.

* 14. 6.
and 1. 6.

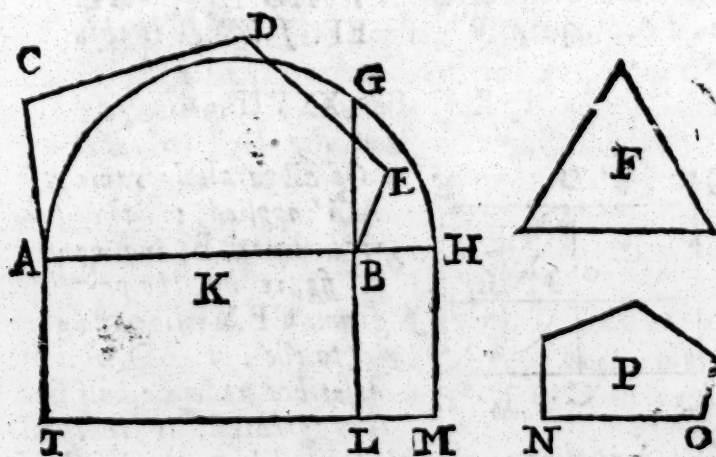
PROP. XXIV.

In every parallelogram ABCD, the parallelograms EG, HF which are about the diameter AC are like to the whole, and also one to the other.



For the Pgrs. EG, HF have each of them one angle common with the whole; a therefore they are equiangular to the whole, and also one to the other. Also both the triangles ABC, AEI, IHC and the triangles ADC, AGI, IFC are equiangular mutually; b therefore AE. b 4. 6. EI :: AB. BC, and b AE. AI :: AB. AC, and b AI. AG :: AC. AD. c Therefore by equality, AE. c 22. 5. AG :: AB. AD. d Therefore the Pgrs. EG, BD d 1. def. 6. are like. After the same manner are HF, BD like also. Therefore, &c.

PROP. XXV.



Unto the right-lined figure given ABEDC to describe another figure P like and alike situate, which also shall be equal to another right-lined figure given F.

a Make the rectangle AL = ABEDC; b also a 45. 1. upon BL make the rectangle BM = F; betwixt b 44. 1. AB and BH c find out a mean proportional NO; c 13. 6. Upon NO d make the polygon P like to the d 18. 6. right-

right lined figure given ABEDC. I say the polygon P so made shall be equal to F that was given.

e cor. 2c. 6.

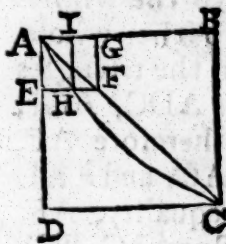
f 1. 6.

g 14. 5.

h constr.

For ABEDC (AL.) $P :: AB. BH :: f AL. BM.$ Therefore $P g = BM h = F.$ Which was to be done.

P R O P. XXVI.



If from the parallelogram ABCD, be taken away another Parallelogram AGFE, like unto the whole, and in like sort set, having also an angle common with it EAG; then is that parallelogram about the same diagonal with the whole, viz. AC.

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI. be drawn parallel to AE. Then are Pgrs. EI, DB, a like. b therefore $AE. EH : AD. DC :: c AE. EF.$ and d consequently $EH = EF.$ f Which is absurd.

a 24. 6.

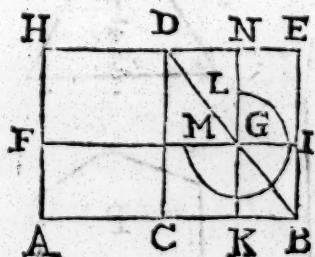
b 1. def. 6.

c hyp.

d 9. 5.

f 9. ax.

P R O P. XXVII.



Of all parallelograms AD, AG applied to the same right line AB, and wanting in figure by the parallelograms CE, KI like and alike set to the Pgr. AD, which is described upon the half line, the greatest is that AD

which is applied to the half being like to the defect KI.

For because that $GE a = GC$, and KI added in common. b thence is $KE = CI c = AM.$ add CG in common, d then is $AG =$ to the Gnomon MBL e but the Gnomon $MBL \sqsupset CE (AD.)$ Therefore $AG \sqsupset AD.$ Which was to be demonstrated.

a 43. 1.

b 2. ax.

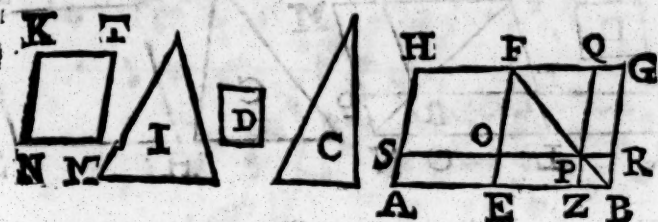
c 36. 1.

d 2. ax.

e 9. ax.

P R O P.

P R O P. XXVIII.

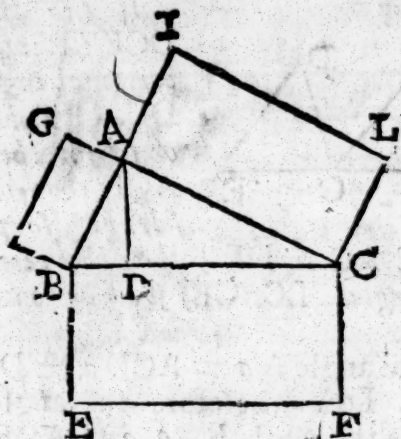


Upon a right line given AB, to apply a parallelo-
 gram AP equal to a right-lined figure given C, deficient
 by a parallelogram ZR which is like to another paral-
 lelogram given D. * Now it is requisit that the right-
 lined figure given C, whereunto the Pgr. to be applied AP * 27. 6.
 must be equal, be not greater than the Pgr. AF which
 is applied upon the half line, the defects being like,
 namely the defect of the Pgr. AF, which is applied to
 the half line, and the defect of the Pgr. D to be applied
 whose defect is to be like to the Pgr. given.

Bisect AB in E; upon EB *a* make the Pgr. EG *a* 18. 6.
 like to the Pgr. D; and *b* let EG = C + I. *c* Make *b* sch. 45. 1.
 the Pgr. NT = I, and like to the Pgr. given D, *c* 25. 6.
 or EG; Draw the diameter FB; Make FO = KN,
 and FQ = KT; thro' O and Q draw the parallels
 SR, QZ. Then is the Pgr. AP that which was
 sought.

For the Pgrs. D, EG, OQ, NT, ZR are all *d* constr. *E*
d like one to the other, and the Pgr. EG = *e* NT 24. 6.
 + C = *e* OQ + C. *f* wherefore C = to the Gnomon *e* constr.
 OBQ *g* = AO + PG = *h* AO + EP = AP. *f* 3. ax.
 Which was to be done. *g* 2. ax.
h 43. 1.

PROP. XXXI.



In right-angled triangles BAC, any figure BF described upon the side BC subtending the right angle BAC, is equal to the figures BG, AL, described upon the sides BA, AC, containing the right angle, like and alike situate to the former BF.

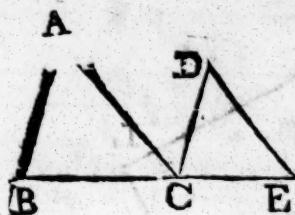
From the right angle BAC let down a perpendicular AD. Because that DC. CA :: a CA. CB, a cor. 8. 6. b therefore AL. BF :: DC. CB. Also, because DB. BA :: a BA. BC, b therefore BG. BF :: DB. BC. c therefore AL + BG. BF :: DC + DB (BC.) BC. c 24. 5. d Therefore AL + BG = BF. Which was to be dem. d sch. 14. 5.

Or thus : BG.BF :: e BAq.BCq. And e AL.BF e 22. 6. :: ACq. BCq. f therefore BG + AL. BF :: BAq. f 24. 5. + ACq. BCq. g Therefore whereas BAq + ACq g sch. 14. 5. = h BCq. h thence is BG + AL = BF. Which was h 47. 1. to be demonstrated.

Coroll.

From this proposition you may learn how to add or subtract any like figures, by the same method that is used in adding and subtracting of squares, in Schol. 47. 1.

PROP.



If two triangles ABC , DCE having two sides proportional to two ($AB. AC :: DC. DE.$) be so compounded or set together at one angle ACD that their homologous sides be also parallel (AB to DC , and AC to $DE.$) then the remaining sides of those triangles $BC. CE$ shall be found placed in one strait line.

a 29. I.

b hyp.

c 6. 6.

d 2. ax.

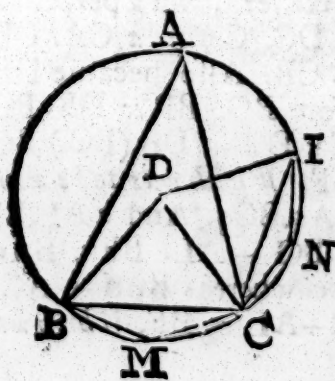
e 32. I.

f 1. ax.

g 14. I.

For the angle $A = ACD = D$, and $AB.AC :: DC. DE.$ therefore the angle $B = DCE$. Therefore the angle $B + A = ACE$. But the angle $B + A + ACB = 2$ right. therefore the angle $ACE + ACB = 2$ right. therefore BCE is a right line, Which was to be demonstrated.

P R O P. XXXIII.



In equal circles $DBCA$, $HFGP$, the angles BDC , FHG have the same ratio with the peripheries BC, FG on which they insist; whether the angles be set at the centers (as BDC, FHG) or at the circumferences, A, E : And in like sort are the Sectors BDC, FHG , because described upon the centers.

Draw

Draw the right lines BC, FG. Make $CI = CB'$ and $GL = FG = LP$. and join DI, HL, HP.

The arch BC $a = CI$. a also the arch FG, GL, a 28. 3. LP are equal ; b therefore the angle $BDC = CDI$ b 27. 3. b and the angle $FGH = GHL = LHP$. Therefore the arch BI is as multiplex of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch FP as multiplex of the arch FG, as the angle FHP is of the angle FHG. But if the arch BI $\square, =, \rightrightarrows$ FP, c then likewise is the c 27. 3. angle BDI $\square, =, \rightrightarrows$ FHP. Therefore is the arch d 6. def. 5.

BDC. FHG

BC. FG :: d the angle BDC. FHG $e :: \frac{\text{---}}{2} \frac{\text{---}}{2}$ e 15. 5.

$f :: A. E.$ Which was to be demonstrated.

Moreover, the angle BMC $g = CNI$; h and f 20. 3. therefore the sement BCM = CIN. k Also the tri- g 27. 3. angle BDC = CDI ; l wherefore the sector BDCM h 24. 3. = CDIN. After the same manner are the sectors k 4. 1. FHG, GHL, LHP equal one to the other. There- l 2. ax. fore since accordingly as the arch BI $\square, =, \rightrightarrows$ FGP. so is likewise the the sector BDI $\square, =, \rightrightarrows$ FHP ; m thence shall be the sector BDC. FHG :: m 6. def. 5. the arch BC, FG. Which was to be demonstrated.

Coroll.

1. Hence, As sector is to sector, so is angle to an- 11. 5. gle.

2. The angle BDC in the center is to four right angles, as the arch BC, on which it consists, to the whole circumference.

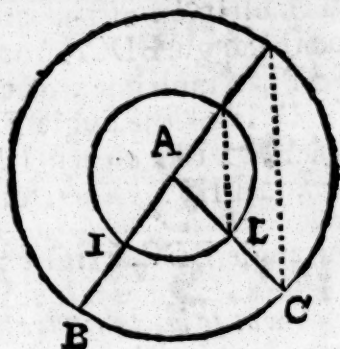
• For as the angle BDC is to a right angle, so is the arch BC to a quadrant. Therefore BDC is so to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Also, the angle A. 2 right :: the arch BC. periphery.

3. Hence, The arches IL, BC of unequal circles which subtend equal angles, whether at the centers, as IAL and BAC, or at the periphery, are like.

For

The sixth Book of, &c.

For IL. periph $::$ angle IAL (BAC) $\frac{1}{4}$ right. Al.
 so Arch BC. periph $::$ angle BAC. $\frac{1}{4}$ right. There-
 fore IL. periph $::$ BC periph. And consequently
 the arches IL and BC are like. Whence



4. Two semidiameters AB, AC cut off like arches
 IL, BC from concentric peripheries.

The End of the sixth Book.

THE

The SEVENTH BOOK
OF
EUCLIDE'S
ELEMENTS.

Definitions.

I. **U**Nity is that, by which every thing that is, is called One.

II. Number is a multitude composed of unities.

III. One number is a part of another, the lesser of the greater, when the lesser measureth the greater.

Every part is denominated from that number, by which it measures the number whereof it is a part; as 4 is called the third part of 12, because it measures 12 by 3.

IV. But the lesser number is termed Parts, when it measures not the greater.

All parts whatsoever are denominated from those two numbers, by which the greatest common measure of the two numbers measures each of them; as 10 is said to be $\frac{2}{3}$ of the number 15; because the greatest common measure, which is 5, measures 10 by 2, and 15 by 3.

V. A number is Multiplex (or Manifold) a greater in comparison of a lesser, when the lesser measureth the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or that which differeth from an even number by an unity.

VIII. A number evenly even is that which an even number measureth by an even number.

IX. But

IX. But a number evenly odd is that which an even number measureth by an odd number.

X. A number oddly odd is that which an odd number measureth by an odd number.

XI. A prime (or first) number is that which is measured only by an unity.

XII. Numbers prime the one to the other, are such as only an unity doth measure, being their common measure.

XIII. A composed number is that which some certain number measureth.

XIV. Numbers composed the one to the other, are they, which some number, being a common measure to them both, doth measure.

In this, and the preceding definition, unity is not a number.

XV. One number is said to multiply another when the number multiplied is so often added to it self, as there are unities in the number multiplying, and another number is produced.

Hence in every multiplication a unity is to the multiplier, as the multiplicand is to the product.

Obf. That many times, when any numbers are to be multiplied (as A into B) the conjunction of the letters denotes the product: So $AB = A \times B$, and $CDE = C \times D \times E$.

XVI. When two numbers multiplying themselves produce another, the number produced is called a plain number; and the numbers which multiplied one another, are called the sides of that number: So $2 (C) \times 3 (D) = 6 = CD$ is a plain number.

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a solid number; and the numbers multiplying one another, are the sides thereof: So $2 (C) \times 3 (D) \times 5 (E) = 30 = CDE$ is a solid number.

XVIII. A square number is that which is equally equal; or, which is contained under two equal

equal numbers. Let *A* be the side of a square; the square is thus noted, *AA*, or *Aq*.

XIX. A Cube is that number which is equally equal equally; or, which is contained under three equal numbers. Let *A* be the side of a Cube; the Cube is thus noted, *AAA*, or *Ac*.

In this definition, and the three foregoing, unity is number.

XX. Numbers are proportional, when the first is as multiplex of the second as the third is of the fourth; or, the same part; or, when a part of the first number measures the second, and the same part of the third measures the fourth, equally: and on the contrary. So *A. B.:: C. D.* that is, 3.9::5.15.

XXI. Like plain, and solid numbers are they which have their sides proportional: Namely, not all the sides, but some.

XXII. A perfect number is that which is equal to its own parts.

As 6. and 28. But a number that is less than it's parts is called an Abounding number; and a greater a Diminutive: so 12 is an abounding, 15 a diminutive number.

XXIII. One number is said to measure another, by that number, which, when it multiplies, or is multiplied by it, it produceth.

In Division, a unity is to the quotient as the divisor is to the dividend. Note, that a number placed under another with a line between them, signifies division:

So $\frac{A}{B} = A \text{ divided by } B$, and $\frac{CA}{B} = C \times A \text{ divided by } B$.

Two numbers are called Terms or Roots of Proportion, lesser than which cannot be found in the same proportion.

Postulates, or Petitions.

1. **T**hat numbers equal or manifold to any number may be taken at pleasure.
2. That a greater number may be taken than any number whatsoever.
3. That

3. That Addition, Substraction, Multiplication, Division, and the Extractions of roots or sides of square and cube numbers be also granted as possible.

Axiomes.

1. **W**Hatsoever agrees with one of many equal numbers, agrees likewise with the rest;

2. Those parts that are the same to the same part, or parts, are the same among themselves.

3. Numbers that are the same parts of equal numbers, or of the same number, are equal amongst themselves.

4. Those numbers, of whom the same number or equal numbers, are the same parts, are equal amongst themselves.

5. Unity measures every number by the units that are in it; that is, by the same number.

6. Every number measures it self by a unity.

7. If one number, multiplying another, produce a third, the multiplier shall measure the product by the multiplied; and the multiplied shall measure the same by the multiplier.

Hence, No prime number is either a plane, solid, square, or cube number.

8. If one number measure another, that number by which it measureth shall measure the same by the unities that are in the number measuring, that is, by the number it self that measures.

9. If a number measuring another, multiply that by which it measureth, or be multiplied by it, it produceth the number which it measureth.

10. How many numbers soever any number measureth, it likewise measures the number composed of them.

11. If a number measure any number it also measureth every number which the said number measureth.

12. A number that measures the whole and a part taken away, doth also measure the residue.

P R O P.

P R O P. I.

If two unequal numbers AB, CD, being given, the lesser CD be continually taken from the greater AB (and the residue EB from CD, &c.) by an alternate subtraction, and the number remaining do never measure the precedent, till the unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

A.....E..G..B	8	5	3
C...F..D	5	3	2
H...			

If you deny it, let AB, CD have a common measure, namely the number H. therefore H measuring CD, doth also measure AE; and consequently the remainder EB; a therefore it likewise measures CF, and b so the remainder FD; a wherefore it also measures EG. But it measured the whole EB, and b therefore it must measure that which remaineth GB, a unity, it self being a number. c Which is absurd. e 9. ax. 1.

P R O P. II.

Two numbers AB, CD being given, not prime the one to the other, to find out their greatest common measure FD.

A.....E.....B	15	9	6
C.....F...D	6	3	
G---			

Take the lesser number CD from the greater AB as often as you can. If nothing remains, a it is manifest that CD is the greatest common measure. But if there remains something (as EB) then take it out of CD, and the residue FD out of EB, and so forward till some number (FD) measure the said EB (b for this will be, before you come to a unity.) FD shall be the greatest common measure. b 1. 7.

For FD c measures EB, and d therefore also CF; c c onstr. and e consequently the whole CD; d therefore likewise AE; and so measures the whole AB. Wherefore e 12. ax. 7.

K ——— fore

fore it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G ;) then whereas G measureth CD , it d must likewise measure AE , e and the residue EB , d as al -
 d 11. ax. 7. g supposed. so CF , e and by consequence the residue FD , g the
 e 12. ax. 7. greater the less. Which is absurd.
 h 9. ax. 1.

Coroll.

Hence, A number that measures two numbers, does also measure their greatest common measure.

PROP. III.

A 12 Three numbers being given, A, B,
 B 8 C, not prime one to another, to find
 D ... 4 out their greatest common measure
 C 6 E.
 E .. 2 Find out D the greatest common
 F --- measure of the two numbers A, B.
 If D measures C the third, it is

clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Proposition preceding. Therefore let E be the greatest common measure of the the said number D and C, and it appears to be the number required.

a const. For E a measures C and D, and D measures A
 b 11. ax. 7. and B; therefore b E measures each of the numbers
 A, B, C: neither shall any greater (F) measure
 c cor. 1. 7. them; for if you affirm that, c then F measuring
 A and B, does likewise measure D their greatest
 common measure; and in like manner, F measuring
 D and C, does also measure E c their greatest com-
 d supposed. mon measure, d the greater the less. e Which
 e 9. ax. 1. is absurd.

Coroll.

Hence, A number that measures three numbers, does also measure their greatest common measure.

PROP.

PROP. IV.

Every less number A is of every greater B either a part or parts. A 6 B 7

If A and B be prime to one another, a A shall be as many parts of the number B, as there are unities in A (as $6 = \frac{6}{7}$.) But if A measures B, it is b plain that A is a part of B (as $6 = \frac{1}{3}$ 18.) b 3. def. 7. Lastly, if A and B be otherwise composed to one another, c the greatest common measure shall de- c 4. def. 7. termine how many parts A does contain of B; as $6 = \frac{2}{3}$ 9.

PROP. V.

A 6 D 4
6 6 4 4
B G C 12. E ... H F 8.

If a number A be a part of a number BC, and another number D the same part of another number EF; then both the numbers together (A + D) shall be the same part of both the numbers together (BC + EF,) which one number A is of one number BC.

For if BC be resolved into its parts BG, GC, equal to A; and EF also into its parts EH, HF equal to D; a the number of parts in BC shall be a hyp. equal to the number of parts in EF. Therefore since $A + D = BG + EH = GC + HF$, thence $A + D$ b const. and shall be as often in $BC + EF$, as A is in BC. Which 2. ax. 1. was to be demonstrated.

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$. then $2a = x$, c 2. ax. 1. and $2b = y$, wherefore $2a + 2b = x + y$. therefore $a + b = \frac{x + y}{2}$. Which was to be demonstrated.

K 2

PROP.

P R O P. VI.

$\begin{array}{rcl} 3 & 3 & 4 & 4 \\ A \dots G \dots B & 6 & D \dots H \dots E & 8 \\ C \dots \dots 9 & & F \dots \dots \dots 12 & \end{array}$

If a number AB be parts of a number C, and another number DE the same parts of another number F; then both numbers together AB + DE shall be of both numbers together C + F the same parts, that one number AB is of one number C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE is equal by supposition; wherefore since AG *a* is the same part of the number C, that DH is of the number F; therefore AG + DH *b* shall be the same part of the compounded number C + F, that one number AG is of one number C. *b* In like manner GB + HE is the same part of the said C + F, that one number GB is of one number C.

a hyp. *b* 5. 7. *c* 2. ax. 7. *c* Therefore AB + DE is the same parts of C + F, that AB is of C. Which was to be demonstrated.

Or thus. Let $a = \frac{1}{2} x$, and $b = \frac{1}{2} y$, and $x + y = g$. then, because $\frac{1}{2} a = \frac{1}{4} x$, and $\frac{1}{2} b = \frac{1}{4} y$, is $\frac{1}{2} a + \frac{1}{2} b = \frac{1}{4} x + \frac{1}{4} y = \frac{1}{4} g$. therefore $a + b = \frac{1}{2} g = \frac{1}{2} x + y$.

P R O P. VII.

$\begin{array}{rcl} 5 & 3 & \\ A \dots E \dots B & 8 & \\ 6 & 10 & 6 \\ G \dots C \dots \dots F \dots D & 16 & \end{array}$

If a number AB be the same part of a number CD, that a part taken away AE is of a part taken away CF; then shall the residue EB be the same part of the residue FD that the whole AB is of the whole CD.

a 1. post. 7. *a* Let EB be the same part of the number GC that AB is of CD, or AE of CF. *b* therefore AE + EB is the same part of CF + GC that AE is of CF, or AB of CD. *c* therefore GF = CD. Take away CF common to both, and *d* there remains *e* 2. ax. 7. GC = FD. *e* Wherefore EB is the same part of the residue FD (GC) that the whole AB is of the whole CB. Which was to be demonstrated.

P R O P.

Or thus. Let $a+b=x$; and $c+d=y$; and $x=3y$, in like manner as $a=3c$; I say $b=3d$. For $3c+3d=3y=x$; $g=a+b$. take away from both f 1. 2. $3c$ $g=a$, and b there remains $3d=b$. Which was g hyp. to be demonstrated.

P R O P. VIII.

In a number AB $\begin{matrix} 6 & 2 & 4 & 2 & 2 \\ A \dots H & G \dots E & L \dots B \end{matrix}$ 16
be the same parts of a number CD, that a $\begin{matrix} 18 & 6 \\ C \dots F \dots D \end{matrix}$ 24
part taken away AE is a part of a part taken away CF; the residue also EB shall be of the residue FD the same parts, that the whole AB is of the whole CD.

Divide AB into AG, GB, parts of the number CD; also AE into AH, HE, parts of the number CF; and take $GL=AH=HE$. a wherefore $HG=EL$. And because b $AG=GB$, c therefore $HG=LB$. Now whereas the whole AG is the same part of the whole CD that the part taken away AH is of the part taken away CF, d the residue HG or EL shall be the same part also of the residue FD that AG is of CD. In like manner, because GB is the same part of the whole CD, that HE or GL are of CF, d therefore the residue LB shall be the same part of the residue FD that GB is of the whole CD. Therefore $EL+LB$ (EB) is the same parts of the residue FD, that the whole AB is of the whole CD. Which was to be demonstrated.

Or thus more easily. Let $a+b=x$, and $c+d=y$. Also $y=2x$ as well as $c=2a$; or, e which is the same, $3y=2x$; and $3c=2a$. I say $d=2b$. For $3c+3d=3y=2x$; f 1. 2. $3c+3d=2a+2b$. g There-fore $3c+3d=2a+2b$. take away from each $3c$ g 1. ax. 1. $3d=2a+2b$, and k there remains $3d=2b$. h therefore $d=2b$. Which was to be demonstrated, k 3 ax. 1. l 8. ax. 7.

The seventh Book of

P R O P. IX.

A ... 4 If a number A be a part of a num.
 4 ber BC, and another number D the
 B ... G ... C 8 same part of another number EF;
 5 D ... 5 then alternately what part or parts
 5 the first A is of the third D, the same
 E ... H ... F 10 part or parts shall the second BC be
 of the fourth EF.

A is supposed $\sqsubset$ D. therefore let BG, GC, and
 EH, HF, parts of the numbers BC, EF be equal;
 BG and GC to A; and EH, HF to D. The mul-
 titude of parts is put equal in both. But it is clear
 a 1. ax. 7. that BG is *a* the same part or parts of EH, that
 and 4. 7. GC is of HF; *b* wherefore BC (BG + GC) is the
 b 5. & 6. 7. same part or parts of EF (EH + HF) that BG alone
 (A) is of EH alone (D.) Which was to be demon.

Or thus. Let $a = \frac{b}{3}$, and $c = \frac{d}{3}$; or $3a = b$, and
 * 15. 5. $3c = d$. then is $\frac{c}{a} = \frac{3c}{3a} = \frac{d}{b}$.

P R O P. X.

A .. G .. B 4 If a number AB be parts of a
 C 6 member C, and another number
 5 DE the same parts of another num-
 D H E 10 ber F; then alternately, what
 F 15 parts or part the first AB is of the
 third DE, the same parts or part
 shall the second C be of the fourth F.

AB is taken $\sqsubset$ DE, and C $\sqsubset$ F. Let AG, GB,
 and DH, HE be parts of the numbers C and F, viz.
 as many in AB, as in DE. It is manifest that AG
 a 9. 7. is the same part of C, that DH is of F. *a* whence
 alternately AG is of DH, and likewise GB of HE
 b 5. & 9. 7. and *b* so conjointly AB of DE the same part, or
 parts, that C is of F. Which was to be demonstrated.

Or thus. Let $a = \frac{2b}{3}$, and $c = \frac{2d}{3}$; or $3a = 2b$
 and $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

P R O P.

PROP. XI.

If a part taken away AE be
to a part taken away CF, as
the whole AB is to the whole
CD, the residue also EB shall be
to the residue FD; as the whole
AB is to the whole CD.

4 3
A....E...B 7
8 6
C.....F.....D 14

First, let AB be $\supset$ CD; a then AB is either a 4. 7.
a part or parts of the number CD; and likewise
AE is b the same part or parts of CF; c therefore b 20. def. 7.
the residue EB is the same part or parts of the re- c 7. or 8. 7.
sidue FD that the whole AB is of the whole CD.
b and so AB. CD :: EB. FD. But if AB be $\subset$ CD,
then according to what is already shewn, will
CD. AB :: FD. EB. therefore by inversion AB.
CD :: EB. FD.

PROP. XII.

If there be divers numbers, A, 4. C, 2. E, 3.
how many soever, proportional B, 8. D, 4. F 6.
(A. B :: C. D :: E. F;) then
as one of the antecedents A is to one of the consequents
B, so shall all the antecedents (A + C + E) be to all the
consequents (B + D + F.)

First, let A, C, E, be $\supset$ B, D, F; then (because
of the same proportions) a shall A be the same a 20. def. 7.
part or parts of B that C is of D; b and likewise b 5, 6, 7.
conjointly A + C shall be the same part or parts
of B + D that A alone is of B alone. In the like
manner A + C + E is the same part or parts of B +
D + F that A is of B. c Therefore A + C + E. B + c 20. def. 7.
D + F :: A. B. But if A, C, E, be put greater than
B, D, F, the same may be shewn by inversion.

The seventh Book of

P R O P. XIII.

A, 3. C, 4. If there be four numbers proportional (A. B :: C. D.) then alternately they shall also be proportional, (A. C :: B. D.)
 B, 9. D, 12.

First, let A and C be $\sqsupset$ B and D, and A $\sqsupset$ C.
 a 20. def. 7. By reason of the same proportion a shall A be the
 b 2. & 10. same part or parts of B, that C is of D. b Therefore alternately A is the same part or parts of C that B is of D. and so A. C :: B. D. But if A be $\sqsubset$ C, and A and C supposed $\sqsupset$ B and D, the matter will be the same by inverting the proportions.

P R O P. XIV.

A, 9. D, 6. If there be numbers, how many soever,
 B, 6. E, 4. A, B, C, and as many more equal to
 C, 3. F, 2. them in multitude, which may be compared two and two in the same proportion (A. B :: D. E. and B. C :: E. F;) they shall also, of equality, be in the same proportion (A. C :: D. F.)

a 13. 7.

For because A. B :: D. E. a therefore alternately is A. D :: B. E :: a C. F. a and so again by changing A. C :: D. F. Which was to be demonstrated.

P R O P. XV.

1. D.. If a unit measure any number B,
 B... 3. E..... 6. and another number D do equally measure some other number E; alternately also shall a unit measure the third number D, as often as the second B doth the fourth E.

a 9. 7.

For seeing 1 is the same part of B, that D is of E; a therefore alternately shall 1 be the same part of D, that B is of E. Which was to be dem.

R R O P.

PROP. XVI.

If two numbers A, B, multiplying themselves the one into the other, produce any numbers AB, BA; the numbers produced AB and BA shall be equal the one to the other.

For because $AB = A \times B$, a therefore shall r be as often in A, as B in AB, b and by consequence b 15. 7. alternately r shall be as often in B as A in AB. But for that $BA = B \times A$, a therefore shall r be as often in B, as A in BA. therefore as often as r is in AB, so often is r in BA. and c so $AB = BA$. Which c 4. ax 7. was to be demonstrated.

PROP. XVII.

If a number multiplying two numbers B, C, produce other number AB, AC; the numbers produced of them shall be in the same proportion that the numbers multiplied are. (AB. AC :: B. C.)

For being $AB = A \times B$, a therefore shall r be as often in A as B in AB. a Likewise because $AC = A \times C$, therefore shall r be as often in A, as C in AC. and so also B as often in AB as C in AC. b b 20. def. 7. wherefore B. AB :: C. AC. c and therefore also alternately B. C :: AB. AC. Which was to be dem.

PROP. XVIII.

If two numbers AB, multiplying any number C, produce other numbers AC, BC; the numbers produced of them shall be in the same proportion that the numbers multiplying are (A. B :: AC. BC.)

For a $AC = CA$, and $BC = CB$; so the same C multiplying A and B produceth AC and BC. b there fore A. B :: AC. BC. Which was to be demonstrated.

Schol.

Schol.

Hence is deduced the vulgar manner of reducing fractions ($\frac{1}{3}$) to the same denomination. For multiply 9 both by 3 and 5, and they produce $\frac{3}{1}$ $\frac{5}{1}$; because by this, $3. 5 :: 27. 45$. Likewise multiply 5 by 7 and 9; there arises $\frac{3}{1}$ $\frac{5}{1}$; because $7. 9 :: 35. 45$.

P R O P. XIX.

A, 4. B, 6. C, 8. D, 12. If there be four numbers in proportion (A. B :: C. D) the number produced of the first and fourth (AD) is equal to the number which is produced of the second and third (BC.) And if the number which is produced of the first and fourth (AD) be equal to that produced of the second and third (BC) those four numbers shall be in proportion (A. B :: C. D.)

a 17. 7.

b hyp.

c 18. 7.

d 9. 5.

e hyp.

f 17. 5.

g 17. 7.

h 18. 7.

k 11. 5.

1. Hyp. For AC. AD a :: C. D b :: A. B c :: AC. BC. d therefore AD = BC. Which was to be dem.

2. Hyp. Because e AD = BC, therefore AC. AD f :: AC. BC. But AC. AD g :: C. D. and AC. BC b :: A. B. k therefore C. D :: A. B. Which was to be demonstrated.

P R O P. XX.

A. B. C. If there be three numbers in proportion (A. B :: B. C) the number contained under the extremes (AC) is equal to the square made of the middle (BB.) And if the number contained under the extremes be equal to that (Bq.) produced of the middle, those three numbers shall

be in proportion $\left(\frac{A}{B} :: \frac{B}{C}\right)$

a 1. ax. 7.

b 19. 7.

1. Hyp. For take D = B. a therefore A. B :: D (B.) C. b wherefore AC = BD, a or BB. Which was to be demonstrated.

2. Hyp.

2. Hyp. Because $AC \propto BD$, d therefore $A.B :: c$ hyp
 $D(B).C$. Which was to be demonstrated. d 19. 7.

PROP. XXI.

Numbers AB, CD , $A \dots G \dots B$ 5. $E \dots \dots \dots 10$.
 being the least of all $C \dots H \dots D$ 3. $F \dots \dots \dots 6$.
 that have the same pro-
 portion with them (E, F) do equally measure the num-
 bers E, F , having the same proportion with them; the
 greater AB the greater E , and the lesser CD the les-
 ser F .

For $AB. CD \propto E. F$ therefore alternately a hyp.
 $AB. E :: CD. F$. c therefore AB is the same part b 13. 7.
 or parts of E that CD is of F . Not parts; for if so, c 2. def. 7.
 then let AG, GB be parts of the number E ; and
 CH, HD , parts of the number F . c therefore AG .
 $E :: CH. F$. and by inversion $AG. CH \propto E. F$ d 13. 7.
 $AB. CD$. therefore AB, CD are not the least in e hyp.
 their proportion; which is contrary to the hypo-
 thesis. Therefore, &c.

PROP. XXII.

If there be three numbers A, B, C ; $A, 4$. $D, 12$.
 and other numbers equal to them in $B, 3$. $E, 8$.
 multitude, D, E, F ; which may be $C, 2$. $F, 6$.
 compared two and two in the same
 proportion: and if also the proportion of them be per-
 turbed ($A. B :: E. F$. and $B. C :: D. E$.) then of e-
 quality they shall be in the same proportion ($A. C ::$
 $D. F$.)

For because $A. B \propto E. F$, therefore shall $AF = a$ hyp.
 BE ; and because $B. C \propto D. E$. b therefore $BE = b$ 19. 7.
 CD . c and consequently $AF = CD$. d Wherefore $A. c$ 1. ax. 1.
 $C :: D. F$. Which was to be demonstrated. d 19. 7.

PROP.

P R O P. XXIII.

A, 9. B, 4. Numbers prime the one to the other, A, B, are the least of all numbers that have the same proportion with them.
 C --- D ---
 E ---

If it be possible, let C and D be less than A and B, and in the same proportion; a therefore C measures A equally as D measures B, namely by the same number F; and so C shall be *b* as often in A as 1 is in E; *c* likewise alternately, E as often in A as 1 in C. By the like inference as many times as 1 is in D, so many times shall E be in B. Therefore E measures both A and B; which consequently are not prime the one to the other, contrary to the hypothesis.

a 21. 7.
 b 23. def. 7.
 c 15. 7.

P R O P. XXIV.

A, 9. B, 4. Numbers A, B, being the least of all that have the same proportion with them, are prime the one to the others.
 C ---
 D --- E ---

If it be possible, let A and B have a common measure C; and let the same measure A by D, and B by E; a therefore $CD = A$, b and $CE = B$. b Wherefore $A : B :: D : E$. But D and E are lesser than A and B, as being but parts of them. Therefore A and B are not the least in their proportion, against the Hypothesis.

a 9. ax. 7.
 b 17. 7.

P R O P. XXV.

A, 9. B, 4. If two numbers A, B, be prime the one to the other, the number C measuring one of them A, shall be prime to the other number B.
 C. 3. D - -

For if you affirm any other D to measure the numbers B and C, a then D measuring C does also measure A; and consequently A and B are not prime the one to the other: Which is against the Hypothesis.

a 11. ax. 7.

P R O P.

PROP. XXVI.

A, 5. C, 8. If two numbers, A, B, be
 B, 3. prime to any number C, the
 AB, 15. E ---- number also produced of them
 F ---- AB shall be prime to the same
 C.

If it be possible, let the number E be a common
 measure to AB, and C; and let $\frac{AF}{E}$ be = F; a 9. ax. 7.

thence $AB = EF$; b wherefore also E. A :: B. F. b 19. 7.
 But because A is prime to C, which E measures, c 25. 7.
 c therefore E and A are prime to one another, d d 23. 7.
 and so least in their own proportion, e and conse- e 21. 7.
 quently they must measure B and F; namely F
 shall measure B, and A shall measure F. There-
 fore seeing E measures both B and C, they shall
 not be prime to one another: Contrary to the
 Hypothesis.

PROP. XXVII.

A, 4. B, 5. If two numbers A, B, be prime to
 Aq, 16. one another, that also which is produ-
 D, 4. ced of one of them (Aq) shall be prime
 to the other B.

Take $D = A$; therefore each of D, and A are a 1. ax. 7.
 prime to B. b wherefore AD or Aq is prime to B. b 26. 7.
 Which was to be demonstrated.

PROP. XXVIII.

A, 5. C, 4. If two numbers A, B be prime
 B, 3. D, 2. to two numbers C, D each to either
 AB, 15. CD, 8. of both, the numbers also produced
 by multiplying them AB, CD shall
 be prime to one another.

For being A and B are prime to C, a therefore a 26. 7.
 shall AB also be prime to the same. And by the
 same

b 26. 7. same reason shall AB be prime to D. *b* Therefore AB is prime to CD. Which was to be demonstrated

P R O P. XXIX.

A, 3. B, 2. If two numbers A, B, be prime
Aq, 9. Bq, 4. to one another, and each multiply-
Ac, 27. Bc, 8. ing himself produce another num-
ber Aq, and Bq; then the number
produced of them (Aq, Bq) shall be prime to one ano-
ther. And if the numbers given at first, A, B, multi-
plying the said produced numbers (Aq, Bq) produce o-
thers (Ac, Bc) those numbers also shall be prime to one
another: And this shall ever happen about the extremes.

For because A is prime to B, a therefore Aq
shall be prime to B. and Aq being prime to B,
a 27. 7. a therefore Aq shall be also be prime to Bq. Again,
because A is as well prime to B and Bq, as Aq is
b 28. 7. to the said B and Bq, *b* therefore shall $A \times Aq$, that
is, Ac, be prime to $B \times Bq$, that is, to Bc: And so
forth of the rest.

P R O P. XXX.

8 5 If two numbers AB,
A B C 13. D ---- BC, be prime the one to
the other; then both
added together (AB) shall be prime to either of them
AB, BC. And if both added together AC be prime to
any one of them AB, the numbers also given in the be-
ginning AB, BC, shall be prime to one another.

1. Hyp. For if you would have AC, AB to be
a 12. ax. 7. compoed, let D be the common measure: a this
shall measure the residue BC: And therefore AB,
BC, are not prime to one another; which is
against the Hypothesis.

2. Hyp. AC, AB being taken for prime to one
another, let D be the common measure of AB, BC.
b 10. ax. 7. *b* But seeing that measures the whole AC, there-
fore AC, AB, are not prime to one another; con-
trary to the Hypothesis.

Coroll.

Hence, A number, which being compounded of two, is prime to one of them, is also prime to the other.

PROP. XXXI.

Every prime number A is prime to every number B, which it measures not.

For if any common measure doth measure both, A, B, a then A will not be a prime number; contrary to the Hypothesis. a 11.def.7.

PROP. XXXII.

If two numbers A, B, multiplying one another produce another AB, and some prime number D measure the number produced of them AB; then shall it also measure of those numbers, A or B, which were given at the beginning.

Suppose the number D not to measure the number A, and let $\frac{AB}{D}$ be = E. a then $AB = DE$; b a 9.ax.7. b 19.7. whence $D : A :: B : E$. c But D is prime to A; d c hyp. and therefore D and A are the least in their proportion; e and consequently D measures B as often as A measures E. Which was to be demonstrated. d 23.7. e 21.7.

PROP. XXXIII.

Every composed number A is measured by some prime number B.

Let one or more numbers a measure A, of which a 13.def.7. let the least be B; that shall be a prime number: For if it be said to be composed, then some a lesser number shall measure it, b which shall also consequently measure A. Wherefore B is not the least of them which measure A, contrary to the Hyp. b 11.ax.7.

PROP.

P R O P. XXXIV.

A, 9. Every number A, is either a prime or measured by some prime number.

For A is necessarily either a prime or a composed number. If it be a prime, 'tis that we affirm. *a* 33. 7. If composed, *a* then some prime number measureth it. Which was to be demonstrated.

P R O P. XXXV.

A, 6. B, 4. C, 8.

D, 2.

H -- I -- K ----

E, 3. F, 2. G, 4.

L ---

How many numbers soever A B C, being given to find the least numbers E, F, G, that have the same proportion with them.

a 23. 7. If A, B, C, be prime to one another, *a* they shall be the least in their proportion. If they be composed, *b* let their greatest common measure be D, which let measure them by E, F, G. These are then the least in the proportion A, B, C.

c 9. *ax.* 7. For $D \times E, F, G$, *c* produceth ABC, *d* therefore they are all in the same proportion. But allow other numbers H, I, K to be the least in the same proportion; *e* which shall therefore equally measure A, B, C, namely by the number L. *f* therefore $L \times H, I, K$, shall produce A, B, C, *g* and consequently $ED = A = HL$. *b* from whence $E : H :: L : D$. But $E \nmid K$; *l* therefore $L \nmid D$, and so D is not the greatest common measure of A, B, C. *h* 19. 7. *k* *suppos.* *l* 20. *def.* 7. Which is against the Hypothesis.

Coroll.

Hence, The greatest common measure of how many numbers soever does measure them by the numbers which are least of all that have the same proportion with them. Whereby appears the vulgar method of reducing fractions to the least terms.

P R O P.

P R O P. XXXVI.

Two numbers being given, A, B, to find out the least number which they measure.

1. *Case*. If A and B be prime the one to the other, AB is the number required. For it is manifest that A and B measure AB. If it be possible, let A and B measure some other number $D \sqsubset AB$, if you please by E, and F. *a* therefore $AE = D = BF$, *b* and so $A.B :: F.E$. But because A and B are prime the one to the other, *d* and so least in their proportion, A shall equally measure F as B does E. But $B.E f :: B.AE (D) g$ Therefore AB shall also measure D, which is less than itself. Which is absurd.

a 9. ax. 7.
b 19. 7.
c hyp.
d 23. 7.
e 21. 7.
f 17. 7.
g 20. def. 7.

2. *Case*. But if A and B be composed one to another, *b* let there be found C and D the least in the same proportion. *k* therefore $AD = BC$; and AD or BC shall be the number sought for.

A, 6. *B*, 4. *F* ---
C, 3. *D*, 2. *G* --- *H* ---
AD, 12.
h 35. 7.

For it is *l* plain that B and D do measure AD or BC. Conceive A and B to measure F $\sqsubset AD$, namely A by G, and B by H. *m* therefore $AG = F = BH$. *n* whence $A.B :: H.G o :: C.D p$ and consequently C equally measures H as D does G. But $D.G q :: AD.AG (F.)$ therefore AD *r* measures F, the greater the less. Which is absurd.

l 7. ax. 7.
m 9. ax. 7.
n 19. 7.
o constr.
p 21. 7.
q 17. 7.
r 20. def. 7.

Coroll.

Hence, If two numbers multiply the least that are in the same proportion, the greater the less, and the less the greater, the least number which they measure shall be produced.

The seventh Book of

P R O P. XXXVII.

A, 2. B, 3. *If two numbers A, B, measure
E 6 any number CD, the least number
C - - - F - - - D which they measure E shall also
measure the same CD.*

a hyp.
b constr.

c 11. ax. 7.

d 12. ax. 7.

If you deny it, take E from CD as often as you can, and leave FD $\neg$ E. therefore seeing A and B a measure E, b and E measures CF, c likewise A and B will measure CF. But a they measure the whole CD; d therefore also they measure the residue FD; and consequently E is not the least which A and B measure: *Contrary to the Hypothesis.*

P R O P. XXXVIII.

A, 3. B, 4. C, 6. *Three numbers being given
D, 12. A, B, C, to find out the least
which they measure.*

a 36. 7.

a Find D to be the least that two of them A and B do measure; which if the third C do also measure, it is manifest that D is the number sought for. But if C do not measure D, let E be the least that C and D do measure. E shall be the number required.

A, 2. B, 3. C, 4. *For it appears by the 11.
D, 6. E, 12. ax. 7. that A, B, C measure
F - - - E; and it is easily shewn
that they measure no other*

b 37. 7.

less than F. For if you affirm they do, b then D measures F, b and consequently E measures the same F, the greater the less. *Which is absurd.*

Coroll.

Hence it appears, that if three numbers measure any number, the least also, which they measure, shall measure the same.

P R O P.

PROP. XXXIX.

If any number B measure a number A, 12.
A, the number measured A, shall have B, 4. C, 3.
a part C denominated of the number
measuring B.

For becau^{se} $\frac{A}{B} a = C$, b shall $A = BC$. c there- a hyp.
fore $\frac{A}{C} = B$. Which was to be demonstrated. b 9. ax. 7.
c 7. ax. 7.

PROP. XL.

If a number A have any part what- A, 15.
soever B, the number C, from which B, 3. C, 5.
the part B is denominated, shall mea-
sure the same.

For being $BC a = A$, b thence $\frac{A}{C} = B$. Which a hyp. and
was to be demonstrated. 9. ax. 7.
b 7. ax. 7.

PROP. XLI.

To find out a number G, which being $\frac{1}{2}$ G, 12.
the least, containeth the parts given $\frac{1}{3}$ H ---
 $\frac{1}{3}, \frac{1}{3}, \frac{1}{4}$ $\frac{1}{4}$

a Let G be found the least which the denomi- a 38. 7.
nators 2, 3, 4, measure; b it is evident that G b 39. 7.
has the parts $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. If it be possible let $H \sqsubset G$ have
the same parts; c therefore 2, 3, 4, measure H; c 40. 7.
and so G is not the least which 2, 3, 4, measure:
against the constr.

The End of the seventh Book.

The EIGHTH BOOK
OF
E U C L I D E ' S
E L E M E N T S.

P R O P. I.

A, 8. B, 12. C, 18. D, 27.
E - F - - G - - - H - - -

I*F there be divers numbers how many soever in continual proportion, A, B, C, D, and their extremes A, D, prime to one another ; then those numbers A, B, C, D, are the least of all numbers that have the same proportion with them.*

a 14. 7.

b 23. 7.

c 21. 7.

For, if it be possible, let there be as many others E, F, G, H, less than A, B, C, D, and in the same proportion with them. *a* Therefore of equality $A. D :: E. H.$ and consequently A and D are prime numbers, *b* and so the least in their proportion, *c* equally measuring E and H, which are less than themselves. *Which is absurd.*

P R O P. II.

A, 2. B, 3.

Aq, 4. AB, 6. Bq, 9.

Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the least numbers continually proportional, as many as shall be required, in the proportion given of A to B.

Let A and B be the least in the proportion given ; Then Aq, AB, Bq, shall be the three least in the same continual proportion that A is to B.

For

For AA. AB $a :: A. B a :: AB. BB$. Likewise a 17. 7. because A and B are prime one to another, c shall b 24. 7. Aq, Bq, be also prime to one another, d and so Aq, c 29. 7. AB, Bq, are $::$ the least in the proportion of A to d 1. 8. B.

Moreover, I say Aq, AqB, ABq, Bc, are the four least in the proportion of A to B. For AqA. AqB $e :: A. B. e :: ABA (AqB.) ABB. e$ and A.B $:: e$ 17. 7. ABq. BBq (Bc.) Therefore since Ac, and Bc, are f prime to one another, likewise g shall Ac, AqB, f 16. 7. ABq, Bc be the four least $::$ in the proportion of g 1. 8. A to B. In the same manner may you find out as many proportional numbers as you please. Which was to be done.

Coroll.

1. Hence, If three numbers, being the least, are proportional, their extremes shall be squares; if four, cubes.

2. How many extremes proportional soever there be, being by this prop. found to be the least in the given proportion, they are prime to one another.

3. Two numbers, being the least in the given proportion, do measure all the mean numbers whatsoever of the least in the same proportion; because they arise from the multiplication of them into certain other numbers.

4. Hence also it appears by the construction, that the series of numbers 1, A, Aq, Ac; 1, B, Bq, Bc; Ac, AqB, ABq, BC, consists of an equal multitude of numbers; and consequently, the extreme numbers of how many soever the least continually proportionals are the last of as many other continually proportionals from a unit, as the extremes Ac, Bc, of the continually proportionals Ac, AqB, ABq, Bc, are the least of as many proportionals from a unit. 1, A, Aq, Ac; and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq are $::$ in the proportion of 1 to A. Also b, Bq, Bc; and A, AB, ABq; and Aq, AqB are $::$ in the proportion of 1 to B.

L 3

P R O P.

P R O P. III.

A, 8. B, 12. C, 18. D, 27. *If there be numbers continually proportional, how many soever, A, B, C, D, being also the least of all that have the same proportion with them; their extremes A, D, are prime to one another.*

a 2. 8. For if there be a found as many numbers the least in the proportion of A to B, they shall be no other than A, B, C, D; therefore, by the second Coroll. of the precedent prop. the extremes A and D are prime to one another. Which was to be demon.

P R O P. IV.

A, 6. B, 5. C, 4. D, 3. *Proportions how many soever being given in the least numbers (A, to H, 4. F, 24. E, 20. G, 15. I -- K -- L -- B, and C to D) to find*

out the least numbers continually proportional in the proportions given.

a 36. 7. a Find out E the least number which B and C
b 3. post. 7. do measure; and let B measure E b as often as A
does another F, viz. by the same number H. b Al
so let C measure the said E as often as D measures
another G. then F, E, G, shall be the least in the
c 9. ax. 7. proportions given. For AH c=F, and BC c=E;
d 18. 7. d therefore A.B::AH.BH e::FE. In like manner
e 7. 5. C.D::E.G; therefore F, E, G, are continually
proportional in the proportions given. And they
are moreover the least in the said proportions; for
conceive other numbers I, K, L, to be the least;
f 21. 7. f then A and B must equally measure I and K, f
and C and D likewise K and L; and so B and C
g 37. 7. measure the same K. g Wherefore also E measures
the same number K, which is less than it self.
Which is absurd.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the least continually in the proportions of A to B, and C to D. Then if E measures I, *b* take another number K which may be equally measur'd by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove than we did in the first part.

h 3. post. 7.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 29. I, 25.
M, 48. L, 40. K, 30. N, 105.

If E do not measure I, let K be the least which E and I do measure; and as often as I measures K, let G as often measure L, and H also M. so likewise let F measure N as often as E measures K. The four numbers M, L, K, N shall be least continually in the given proportions; which we may demonstrate as before.

P R O P. V.

Plain numbers CD, EF, C, 4. E, 3.
are in that proportion to D, 6. F, 16. ED, 18.
one another which is composed of their sides.
 $\frac{CD}{EF} = \frac{CD}{ED} + \frac{ED}{EF}$ CD, 24. EF, 48.

For because CD. ED *a* :: C.E; *a* and ED. EF :: *a* 17. 7.
D.F. and $\frac{CD}{EF} = \frac{CD}{ED} + \frac{ED}{EF}$ *c* then shall be the pro- *b* 20. def. 5.
c 11. 5.

portion $\frac{CD}{EF} = \frac{C}{E} + \frac{D}{F}$ Which was to be demon.

P R O P. VI.

If there be numbers continually proportional how many soever, A, B, C, D, E, and the first A do not measure the second B, neither shall any of the other measure any one of the rest.

L 4

Be-

- a 20. def. 7. Because A does not measure B, a neither shall any one measure that which next follows ; being A.B :: B. C :: C. D, &c. b Take three numbers, F, G, H, the least in the proportion of A to B. therefore since A does not measure B, a neither shall F measure G. c therefore F is not a unit. But F and H are prime one to another ; and so, d being of equality A. C :: F. H. and F does not measure H, a neither shall A measure C ; and consequently neither shall B measure D, nor C measure E, &c. because A. C e :: B. D e :: C. E, &c. In like manner four or five numbers being taken the least in the proportion of A to B, it will appear that A does not measure D and E ; nor does B measure E and F, &c. Wherefore none of them shall measure any other. Which was to be demonstrated.

P R O P. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many soever A, B, C, D, E, and the first A measure the last E, it shall also measure the second B.

- a 6. 7. If you deny that A measures B, a then neither shall it measure E ; Which is contrary to the Hyp.

P R O P. VIII.

A, 24. C, 36. D, 54. B, 81. If between two G, 8. H, 12. I, 18. K, 27. numbers A, B, there E, 32. L, 48. M, 72. F, 108. fall mean proportional numbers in continual

proportion C, D ; as many mean continually proportional numbers as fall between them, so many also mean continually proportional numbers shall fall between two other numbers E, F, which have the same proportion with them. (L, M.)

- a 35. 7. a Take G, H, I, K, the least :: in the proportion
b 14. 7. of A to C ; b of equality shall G K :: A. B c :: E. F.
c hyp. But G, and K d are prime one to another. e Where-
d 3. 8. fore G measures E as often as K does F. Let H
e 21. 7. measure L, and I likewise M by the same number.
f constr. f therefore E, L, M, F, are in such proportion as G,
H, I, K, that is, as A, B, C, D. Which was to be dem.

P R O P.

PROP. IX.

If two numbers A, B be prime to one another, and mean numbers in continual proportion C, D fall between them; as many mean numbers in continual proportion as fall between them, so many means also (E, G; and F, I) shall fall in continual proportion between either of them and a unity.

It is evident, that 1, E, G, A, and 1, F, I, B, are $\therefore$, and as many as A, C, D, B, namely by the 4. Coroll. 2. 8. Which was to be demonstrated.

PROP. X.

If between two numbers A, B, and a unit, numbers continually proportional (E, D, and F, G,) do fall, how many mean numbers in continual proportion fall between either of them and a unity, so many means also shall fall in continual proportion between them, I, K.

For E, DF, G, and A, DqF (I) DG (K) B are $\therefore$ by 2. 8. therefore, &c.

PROP. XI.

Between two square numbers Aq, Bq, there is one mean proportional number AB: and Aq to Bq is in double proportion of that of the side A to the side B.

It is manifest that Aq, AB, Bq, are $\therefore$; b a 17. 7. and consequently also $\frac{Aq}{Bq} = \frac{A}{B}$ doubly. Which b 10. def. 5. was to be demonstrated.

PROP.

P R O P. XII.

Ac, 27. AqB, 36. ABq, 48. Bc, 64.

A, 3. B, 4.

Aq, 9. AB, 12. Bq, 16.

Between two
cube numbers, Ac,
Bc, there are two
mean proportional

numbers AqB, ABq : and the cube Ac is to the cube
Bc in treble proportion of that in which the side A is
to the side B.

a 2. 8.

a For Ac, AqB, ABq, Bc, are $\therefore$ in the propor.

b 10. def. 5. tion of A to B; b and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ trebly.

Which was to be demonstrated.

P R O P. XIII.

A, 2. B, 4. C, 8.

Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.

Ac, 8. AqB, 10. ABq, 32. B., 64. BqC, 128. BCq, 256. Cc, 512.

If there be numbers in continual proportion how many soever A, B, C; and every of them multiplying it self produce certain numbers; the numbers produced of them Aq, Bq, Cq, shall be proportional: And if the numbers first given A, B, C, multiplying their products Aq, Bq, Cq, produce other numbers, Ac, Bc, Cc, they also shall be proportional; and this shall ever happen to the extremes.

For Aq, AB, Bq, BC, Cq a are $\therefore$; b therefore of equality Aq. Bq $\therefore$ Bq. Cq. Which was to be dem.

a 2. 1.

b 14. 7.

a Allo Ac, AqB, ABq, Bc, BqC, BCq, Cc, are $\therefore$; b therefore likewise of equality Ac. Bc $\therefore$ Bc. Cc. Which was to be demonstrated.

P R O P. XIV.

Aq, 4. AB, 12. Bq, 36.

A, 2.

B, 6.

If a square number Aq measure a square number Bq, the side also of the one (A) shall measure the side of the other (B:) and if the side of one square A measure the side of another B, the square Aq shall likewise measure the square Bq.

1. Hyp.

1. Hyp. For Aq. AB $a :: AB$. Bq. therefore seeing by the hypothesis Aq measures Bq, b it shall measure also AB. But Aq. AB $:: A$. B. c therefore also A measures B. Which was to be demonstrated.

2. Hyp. A measures B. c therefore Aq shall as well measure AB, c as AB measures Bq; d consequently Aq measures Bq. Which was to be demon.

P R O P. XV.

If a cube number A, 2. B, 6.
Ac measure a cube Ac, 8. AqB, 24. ABq, 72. Bc, 216.
number Bc, then the
side of the one (A) shall measure the side of the other (B:) And if the side A of one cube Ac measure the side B of the other BC, also the cube Ac shall measure the cube Bc.

1. Hyp. For Ac, AqB, ABq, Bc a are $::$, therefore Ac, b measuring the extreme Bc, shall also c measure the second AqB. But Ac. AqB $:: A$. B. d therefore A shall also measure B.

2. Hyp. A measures B; d therefore Ac measures AqB, which also measures ABq, and that Bc; e therefore Ac shall measure Bc. Which was to be dem.

P R O P. XVI.

If a square number Aq do not A, 4. B, 9.
measure a square number Bq, nei- Aq, 16. Bq, 81.
ther shall the side of the one A measure the side of the other B: And if A the side of the one square Aq do not measure B the side of the other Bq, neither shall the square Aq measure the square Bq.

1. Hyp. For if you affirm that A measures B, a then Aq also shall measure Bq. against the Hyp. a 14. 8.

2. Hyp. If you maintain Aq to measure Bq; a then likewise A shall measure B. contrary to the Hypothesis.

P R O P.

P R O P. XVII.

A, 2. B, 3. If a cube number Ac do not mea-
 Ac, 8. Bc, 27. sure a cube number Bc, neither
 shall the side of one A measure the
 side of the other B: And if A the side of one cube Ac
 do not measure B the side of the other Bc, neither shall
 the cube Ac measure the cube BC.

a 15. 8.

1. Hyp. Let A measure B; a then Ac shall mea-
 sure Bc. against the Hypothesis.

2. Hyp. Let A measure Bc; then A shall mea-
 sure B; which is also against the Hypothesis.

P R O P. XVIII.

C, 6. D, 2.

CD, 12.

E, 9. F, 3. DE, 18.

EF, 27.

Between two like plane num-
 bers CD and EF there is one mean
 proportional number DE: And
 the plane CD is to the plane EF
 in double proportion of that which
 the side C hath to the homologous side (or of like pro-
 portion) E.

* 21. def. 7. Being * by the Hypothesis C.D :: E.F. therefore

a 17. 7. by inversion C. E :: D. F. But C. E a :: CD. DE;

b 11. 5. a and D.F :: DE. EF. b therefore QD.DE::DE.EF.

c 10. def. 5. c Wherefore the proportion of CD to EF is double
 to that of CD to DE, that is, to the proportion
 of C to E, or D to F.

Coroll.

Hence it is apparent, That between two like
 plane numbers there falls one mean proportional
 in the proportion of the homologous sides.

P R O P.

P R O P. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.

CD, 6. DF, 12. FG, 24.

C, 2. D, 3. E, 5. F, 3. G, 6. H, 10.

Between two like solid numbers CDE, FGH, there are two mean proportional numbers DFE, FGE. And the solid CDE is to the solid FGH, in treble proportion of that which the homologous side C has to the homologous side F.

Whereas by the * hyp. $C.D :: F.G$, and $D.E ::$ * 21. def. 7. $G.H$. therefore *a* by inversion shall $C.F :: D.G ::$ *a* 13. 7. $E.H$. But $CD.DF$ *b* $:: C.F$, and $DF.FG$ *b* $:: D.G$; *a* 13. 7. *c* wherefore $CD.DF :: DF.FG :: E.H$. *d* and accordingly $CDE.DFE :: DFE.FGE :: E.H :: FGE.FGH$. *c* 11. 5. *d* 17. 7. Therefore between CDE, FGH, fall two mean proportionals DFE, FGE. *e* And so it is *e* 10. def. 7. plain that the proportion of CDE to FGH is treble to that of CDE to FDE, or C to F. Which was to be demonstrated.

Coroll.

Hereby it is manifest, that between two like solid numbers there fall two mean proportionals in the proportion of the homologous sides.

P R O P. XX.

A, 12. C, 18. B, 27. *If between two numbers A, D, 2. E, 3. F, 6. G, 9. B, there fall one mean proportional number C; those numbers A, B, are like plane numbers.*

a Take D and E the least in the proportion of A *a* 35. 7. to C, or C to B. then D measures A equally as E does C, viz. by the same number F; *b* also D equally *b* 21. 7. measures C, as E does B, viz. by the same number G. *c* Therefore $DF = A$, and $FG = B$. *d* and consequently A and B are plane numbers. But because *c* 9. ax. 7. *d* 6. def. 7. $EF = C$ *e* $= DG$, *e* shall $D.E :: F.G$. and alternately *e* 19. 7. nately

f 21.def.7. nately D.F :: E. G. f Therefore the plane numbers A and B are also like. Which was to be demon.

P R O P. XXI.

A, 16. C, 24. D, 36. B. 54. If between two numbers A, B there fall two mea proportional numbers C, D; those numbers A, B are like solid numbers.

a 2. 8. a Take E, F, G, the least :: in the proportion of
b 10. 8. A to C. b then D and G are like plane numbers: let the sides of this be H and P, and of that K and L. c therefore H. K :: P. L :: d E. F. But E, F, G, do e equally measure A, C, D, viz. by the same number M. and likewise the said numbers E, F, G, do equally measure the numbers C, D, B, viz. by the same number N. f Therefore A = EM = HPM, g 17.def.7. f and B = GN = KLN; g and so A and B are solid numbers. But for that C f = FM, and f = FN, therefore shall M. N b :: FM. FN k :: C. D l :: E. F :: k 7. 5. H. K :: P. L. m wherefore A and B are like solid l constr. numbers. Which was to be demonstrated.
m 21.def.7.

Lemma.

AE, BF, CG, DH, If proportional numbers A, A, B, C, D, B, C, D measure proportional E, F, G, H. numbers AE, BF, CG, DH by the numbers E, F, G, H, these numbers (E, F, G, H) shall be proportional.

a 19. 7. For being AEDH a = BFCG, a and AD = BC,

b 1. ax. 7. bthence will $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ c that is, EH = FG.

c 9. ax. 7. a Therefore E. F :: G. H. Which was to be demon.

Coroll.

d 15.def.7. Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. d For 1.B :: B.Bq d and 1.A ::

e lem. prec. A.Aq, e therof. $\frac{1}{A} \cdot B :: \frac{B.Bq}{AAq}$ d therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$.

In like manner $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$ and so of the rest.

P R O P.

P R O P. XXII.

If three numbers Aq, B, C be continually proportional, and the first Aq a square, the third C shall also be a square.

$Aq, B, C.$
4, 8, 16.

For because $Aq : C :: Bq : b$ thence is $C = \frac{Bq}{Aq}$ a 20. 7.
 $C = Q. \frac{B}{A}.$ But it is plain that $\frac{B}{A}$ is a number, d be- b 7. ax. 7.
cause $\frac{Bq}{Aq}$ or C is a number. Therefore if three, &c. c. cor. of the
lem. prec.
d hyp. and
14. 8.

P R O P. XXIII.

If four numbers A, B, C, D be continually proportional, and the first of them A a cube, the fourth also D shall be a cube.

$A, B, C, D.$
8, 12, 18, 27.

For because $A : D :: B : C$, b therefore $D = \frac{BC}{A}$ a 19. 7.
 $C = \frac{B}{A} \times C$; that is (because $A : C :: Bq : d$ Bq, and c 7. ax. 7.
b thence $C = \frac{Bq}{Ac}$) $D = \frac{B}{Ac} \times \frac{Bq}{Ac} = \frac{Bc}{Acc} = C \times \frac{B}{Ac}$ c cor. of the
prec. lem.
d 20. 7.
But it is evident e that $\frac{B}{Ac}$ is a number, because $\frac{Bc}{Acc}$ e 15. 8.
or D is supposed a number. Therefore if four numbers, &c.

P R O P. XXIV.

If two numbers A, B , be in the same proportion one to another, that a square number C is to a square number D , and the first A be a square number, the second also B shall be a square number.

$A, 16. 24. B, 36.$
 $C, 4. 6. D, 9.$

Between C and D being square numbers, * and * 8. 8.
between A and B having the same proportion, a falls

- a 11. 8. *a* falls one mean proportional. Therefore *b* being
 b hyp. *A* is a square number, *c* *B* also shall be a square
 c 22. 8. number. Which was to be demonstrated.

Coroll.

1. Hence, If there be two like numbers *AB*,
CD ($A.B :: C.D$) and the first *AB* be a square,
 the second also *CD* shall be a square.
 * 12. and * For $AB.CD :: Aq.Cq$.
 18 8. 2. From hence it appears, That the proportion
 of any square number to any other not square can-
 not possibly be declared in two square numbers.
 Whence it cannot be $Q.Q :: 1.2$. nor $1.5 :: Q$.
Q, &c.

P R O P. XXV.

C, 64. 96. 144. *D*, 216. If two numbers *A*, *B*,
A, 8. 12. 18. *B*, 27. be in the same proportion
 one to another, that a cube
 number *C* is to a cube number *D*, the first of them *A*
 being a cube number; the second *B* shall likewise be a
 cube number.

- a 12. 8. *a* Between the cube numbers *C* and *D*, *b* and so
 b 8. 8. between *A* and *B* having the same proportion, fall
 c hyp. two mean proportionals; therefore *c* because *A* is
 d 23. 8. a cube, *d* shall *B* be a cube also. Which was to be
 demonstrated.

Coroll.

1. Hence, If there be two numbers *ABC*, *DEF*
 ($A.B :: D.E$, and $B.C :: E.F$;) and the first *ABC*
 be a cube, the second *DEF* shall be a cube also.
 * 12. and * For $ABC.DEF :: Ac.Dc$.
 19. 8. 2. It is perspicuous from hence, That the pro-
 portion of any cube number to any other number
 not a cube cannot be found in two cube numbers.

P R O P.

PROP. XXVI.

Like plane numbers A , A , 20. C , 30. B , 45.
 B , are in the same proportion one to another, that a square number is in to a square number.

Between A and B a falls one mean proportional a 18. 8.
 number C ; b take three numbers D , E , F the least b 2. 8.
 $\therefore$ in the proportion of A to C , the extremes D ,
 F , b shall be square numbers. But of equality A .
 $Bc :: D. F.$ therefore $A. B :: Q. Q.$ Which was to c 14. 7.
 be demonstrated.

PROP. XXVII.

Like solid numbers A , 16. C , 24. D , 36. B , 54.
 B , are in the same proportion one to another, that a cube number is in to a cube number.

a Between A and B fall two mean proportional a 19. 8.
 numbers, namely C and D : b take four numbers b 2. 8.
 E , F , G , H the least $\therefore$ in the proportion of A to
 C , b the extremes E , H , are cube numbers. But
 $A. B c :: E. H :: C. C.$ Which was to be demon. c 14. 7.
 Schol.

1. From hence is inferred, that no number in See Cla.
 proportion superparticular, superbipartient, or *vins.*
 double, or any other manifold proportion not de-
 nominated from a square number, are like plane
 numbers.

2. Likewise, that neither any two prime num-
 bers, nor any two numbers prime one to another,
 not being squares, can be like plane numbers.

The End of the eighth Book.

(178.)

The NINTH BOOK
O F
EUCLIDE'S
ELEMENTS.

PROPOSITION I.

A, 6. B, 54.

Aq, 36. 108. AB, 324.

If two like plane numbers A, B , multiplying one another, produce a number AB , the number produced AB shall be a square number.

a 17. 7.

b 18. 8.

c 8. 8.

d 22. 8.

For $A. B a :: Aq. AB$; wherefore since one mean proportional b falls between A and B , c likewise one mean proportional number shall fall between Aq and AB : therefore being the first Aq is a square number, d the third AB shall be a square number too. Which was to be demonstrated.

x 19. 7.

y 1. ax. 7.

Or thus. Let $ab\ cd$, be like plane numbers; namely, $a.b :: c.d$. x therefore $ad=bc$. and so likewise $abcd$, or $adbc\ y=ad\ ad=Q: ad$.

PROP. II.

A, 6. B, 54.

Aq, 36. AB, 324.

If two numbers A, B , multiplying one another, produce a square number AB , those numbers A, B are like plane numbers.

a 17. 7.

b 18. 8.

c 8. 8.

d 20. 8.

For $A.B a :: Aq.AB$; wherefore being between $Aq.AB$, b there falls one mean proportional number, c likewise one mean shall fall between A and B . d therefore A and B are like planes. Which was to be demonstrated.

PROP.

PROP. III.

If a cube number A mul. A , 2. Ac , 8. Acc , 64.
 multiplying it self produce a
 number Acc ; the number produced Acc shall be a cube
 number.

For 1. $A a :: A$, $Aq b :: Aq$, Ac . therefore be- a 15. def. 7
 tween 1 and Ac fall two mean proportionals. But b 17. 7.
 1. $Ac a :: Ac$, Acc . c therefore between Ac and c 8. 8.
 Acc , fall also two mean proportionals: and so by
 consequence seeing Ac is a cube, d Acc shall be a d 23. 8.
 cube also. Which was to be demonstrated.

Or thus; aaa (Ac) multiplied into it self makes
 $aaaaaa$ (Acc); this is a cube, whole side is aa .

PROP. IV.

If a cube number Ac mul. Ac , 8. Bc , 27.
 multiplying a cube number Bc Acc , 64. Ac , Bc , 216.
 produce a number $AcBc$, the
 produced number $AcBc$ shall be a cube.

For Ac , Bc , $a :: Acc$, $AcBc$. But between Ac and a 17. 7.
 Bc b two mean proportional numbers fall; c there- b 12. 8.
 fore there fall as many between Acc and $AcBc$. So c 8. 8.
 that whereas Acc is a cube number, d $AcBc$ shall d 23. 8.
 be such also. Which was to be demonstrated.

Or thus; $AcBc = aaabbb$ ($ababab$) $= C^3$: ab .

PROP. V.

If a cube number Ac mul. Ac , 8. B , 27.
 multiplying a number B produce Acc , 64. AcB , 216.
 a cube number AcB ; the num-
 ber multiplied B shall also be a cube.

For Acc , AcB , $a :: Ac$, B . But between Acc and a 17. 7.
 AcB b fall two mean proportionals; c therefore b 12. 8.
 also as many shall fall between Ac and B . whence c 8. 8.
 Ac being a cube number, d B shall be a cube num- d 23. 8.
 ber too. Which was to be demonstrated.

M 2

PROP.

P R O P. VI.

A, 8. Aq, 64. Ac, 512. *If a number A multiplying it self produce a cube*
Aq, that number A it self is a cube.

a hyp. For because Aq a is a cube, and AqA (Ac) b al-
 b 19.def.7. so a cube; therefore c shall A be a cube. Which
 c 5. 9. was to be demonstrated.

P R O P. VII.

A, 6. B, 11. AB, 66. *If a composed number A*
 D, 2. E, 3. *multiplying any number B,*
produce a number AB, the
number produced AB shall be a solid number.

a 13.def.7. Being A is a composed number, a some other
 b 9.ax. 7. number D measures it; conceive by E. b therefore
 c 17.def.7. $A=DE$: c whence $DEB=AB$ is a solid number.
Which was to be demonstrated.

P R O P. VIII.

1. a, 3. a^2 , 9. a^3 , 27. a^4 , 81. a^5 , 242. a^6 , 729.

If from a unit there be numbers continually propor-
tional how many soever (1. a, a^2 , a^3 , a^4 , &c.) the third
number from a unit a^2 is a square number; and so are
all forward, leaving one between (a^4 , a^6 , a^8 , &c.) But
the fourth a^3 is a cube number; and so are all for-
ward, leaving two between (a^6 , a^9 , &c.) The seventh
also a^6 is both a cube number and a square; and so
are all forward, leaving five between (a^{12} , a^{18} , &c.)

For 1. $a^2=Q. a.$ and $a^4=aaaa=Q. aa.$ and a^6
 $=aaaaaa=Q. aaa$, &c.

2. $a^3=aaa=C. a.$ and $a^6=aaaaaa=C. aa.$ and
 $aaaaaaaa=C. aaa$, &c.

3. $a^6=aaaaaa=C. aa=Q. aaa.$ therefore, &c.

a hyp. Or according to Euclide; Because 1. $a : a :: a : a^2$.
 b 26. 7. b shall $a^2=Q. a.$ therefore seeing a^2, a^3, a^4 , are ::,
 c 12. 8. c the third a^4 shall be a square number; and so
 likewise a^6, a^8 , &c. Also because 1. $a : a :: a^2 : a^3$.
 d 23. 8. therefore shall $a^3 b=a^2 \times a=C : a.$ d therefore the
 fourth from a^3 , namely a^6 , shall be likewise a
 cube, &c. and consequently a^6 is both a cube and
 a square number, &c.

P R O P.

PROP. IX.

If from a unit there be numbers $1, 2, 4, 8, 16, 32, 64, 128, 256, \&c.$
 how many soever

continually proportional ($1, 2, 4, 8, 16, 32, 64, 128, 256, \&c.$) and the number following the unit (2) be a square; then all the rest, $4, 8, 16, 32, 64, 128, 256, \&c.$ shall be squares too. But if the number next the unit (2) be a cube, then all the following numbers $8, 16, 32, 64, 128, 256, \&c.$ shall be cube numbers.

1. Hyp. For $2^2, 4^2, 8^2, \&c.$ are square numbers by the prec. prop. also being 2 is taken to be a square, 4 therefore the third 8 shall be a square, and likewise $16, 32, 64, \&c.$ and so all. a 22. 8.

2. Hyp. 2 is taken to be a cube, 8 therefore $2^3, 8^3, 16^3, \&c.$ are cubes: but by the prec. $2^3, 8^3, 16^3, \&c.$ are cubes: lastly, because $1, 2 :: 2, 4$. 4 therefore shall $2^2 = Q$: 2 . but a cube multiplied into it self produces a cube; therefore 2^2 is a cube, 4 and consequently the fourth from it 16 , and in like manner $64, 216, \&c.$ are cubes. therefore all, $\&c.$ b 23. 8.
 Which was to be demonstrated. c 20. 7.
d 3. 9.
e 23. 8.

Peradventure more clearly thus. Let b be the side of the square number a , and so the series $a, a^2, a^3, a^4, \&c.$ will be otherwise expressed, thus, $bb, b^4, b^6, b^8, \&c.$ It is evident that all these numbers are squares, and may be thus expressed, $Q: b, Q: bb: Q: bbb, Q: bbbb, \&c.$

In like manner, if b be the side of the cube a , the series may be expressed thus, $b^3, b^6, b^9, b^{12}, \&c.$ or $C: b, C: b^2, C: b^3, C: b^4, \&c.$

PROP. X.

If from a unit there be numbers $1, 2, 4, 8, 16, 32, 64, 128, 256, \&c.$ how many soever continually proportional ($1, 2, 4, 8, 16, 32, 64, 128, 256, \&c.$) and the number next the unit (2) be not a square number; then is none of the rest following a square number, excepting 4 the third from the unit, and so all forward, leaving one between ($16, 64, 256, \&c.$)

M 3

But

But if that (a) which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, saving a^3 the fourth from the unit, and so all forward, leaving two between, a^6, a^9, a^{12} , &c.

a hyp. **b suppos. &** 1. Hyp. For if it be possible, let a^5 be a square number; therefore because $a \cdot a^2 a :: a^4$, a^5 , and by inversion $a^5 \cdot a^4 :: a^2 \cdot a$; and also a^5 and a^4 square numbers, and the first a^2 a square, c therefore a shall be likewise a square; *contrary to the hyp.*

c 24. 8. 2. Hyp. If it may be, let a^4 be a cube; being **d 24. 7.** d of equality $a^4 \cdot a^6 :: a \cdot a^3$. and inversely $a^6 \cdot a^4 :: a^3 \cdot a$; and also being a^6 and a^4 are cubes, and the first a^3 a cube, e therefore a shall be a cube also; **e 25. 8.** *against the hypothesis.*

P R O P. XI.

1. $a, a^2, a^3, a^4, a^5, a^6$. If there be numbers how
1, 3, 9, 27, 81, 243, 729. many soever in continual
proportion from a unit (1,
 a, a^2, a^3 , &c.) the less measureth the greater by some
one of them that are amongst the proportional numbers.

a 3. ax. 7. Because $1 \cdot a :: a \cdot aa$. a therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$. Also
& 20. def. 7.

b 14. 7. because $1 \cdot aa b :: a \cdot aaa$. a therefore $\frac{aaa}{a} = aa =$

$\frac{a^4}{aa} = \frac{a^5}{a^3}$, &c. Lastly because $1 \cdot a^3 :: b \cdot a \cdot a^4$. therefore

$\frac{a^4}{a} = a^3 = \frac{a^6}{a^3}$, &c.

Coroll.

Hence, If a number that measures any one of proportional numbers, be not one of the said numbers, neither shall the number by which it measures the said proportional numbers, be one of them.

P R O P.

P R O P. XII.

If there be numbers how many soever in continual proportion from a unit ($1, a, a^2, a^3, a^4$.) whatsoever prime numbers B measure the last a^4 , the same (B) shall also measure the number (a) which follows next after the unit.

If you say B does not measure a , a then B is prime to a ; b and also B is prime to a^2 ; c and so consequently to a^4 , which it is supposed to measure. Which is absurd.

Coroll.

1. Therefore every prime number that measures the last, does also measure all those other numbers that precede the last.

2. If any number not measuring that next to the unit, does yet measure the last, it is a composed number.

3. If the number next to the unit be a prime, no other prime number shall measure the last.

P R O P. XIII.

If from a unit be numbers in continual proportion how many soever (a, a^2, a^3 , &c.) and that after a unit (a) a prime; then shall no other measure the greatest number, but those which are amongst the said proportional numbers.

If it be possible, let some other E measure a^4 , viz. by F . a then F shall be some other beside a , a^2, a^3 . But because E measuring a^4 , does not measure a , b therefore E shall be a composed number, c and some prime number measure it, d which does consequently measure a^4 , e and so no other than a . therefore a measures E . So also may F be shewn to be a composed number, measuring a^4 .

M 4

and 9.

f 9. ax. 7. and so that a measures F . Therefore seeing $EF = a^4 = a \times a^3$, g shall $a :: F : a^3$. Consequently, g 19. 7. whereas a measures E , b likewise F shall equally h 20. def. 7. measure a^3 , viz. by the same number G . k Nor k cor. 11. 9. shall G be a , or a^2 . therefore, as before, G is a composed number, and a measures it. Wherefore being that $FG = a^3 = a^2 \times a$, g shall $a :: F : G : a^2$. and so because A measures F , b G shall equally measure a^2 . viz. by the same number H , k which is not a . Therefore being $GH =$ l 23. def. 7. $a^2 = aa$, l thence $H : a :: a : G$. and because a measures G (as before) m 20. def. 7. m H also shall measure a , which is a prime number. Which is impossible.

P R O P. XIV.

A, 30. If certain prime numbers
B, 2. C, 3. D, 5. B, C, D, do measure the least
E -- F -- number A, no other prime number
besides those that measured it at first.

a 9. ax. 7. If it is possible, let $\frac{A}{E}$ be $= F$. a then $A = EF$.

b 32. 7. b therefore every of the prime numbers B, C, D measures one of those E, F. not E. which is taken to be a prime; therefore F, which is less than it self A; contrary to the hypothesis.

P R O P. XV.

A, 9. B, 12. C, 16. If three numbers continually proportional A, B, C, be the least of all that have the same proportion with them; any two of them added together shall be a prime to the third.

a 35. 7. a Take D and E the least in proportion of A to b 2. 8. B; b then $A = Dq$, and b $C = Eq$, b and $B = DE$. But

But because Dc is prime to E , d therefore shall D c 24. 7.
 $+ E$ be prime to both D and E . * therefore $D \times D$ d 30. 7.
 $+ Ee = Dq + DE$ ($f A + B$) is prime to E , and * 26. 7.
 so to C or Eq . Which was to be demonstrated. e 3. 2.

g In like manner $DE + Eq$ ($B + C$) is prime f before.
 to D , and consequently to $A = Dq$. Which was g 27. 7.
 to be demonstrated. h 26. 7.

Lastly, because B h is prime to $D + E$, it shall k 4. 2.
 also be prime to the square of it $k Dq + 2 DE +$ l 30. 7.
 Eq ($A + 2 B + C$); l wherefore the said B shall
 be prime to $A + B + C$, l and so likewise to A
 $+ C$. Which was to be demonstrated.

P R O P. XVI.

If two numbers A, B , be $A, 3. B, 5. C----$
 prime to one another, the se-
 cond B shall not be to any other C , as the first A is to
 the second B .

If you affirm $A. B :: B. C$. then whereas A and
 B a are the least in their proportion, A b shall a 23. 7.
 measure B as many times as B does C ; but A c b 21. 7.
 measures it self also; therefore A and B are not c 6. ax. 7.
 prime to one another. against the Hyp.

P R O P. XVII.

If there be $A, 8. B, 12. C, 18. D, 27. E----$
 numbers how
 many soever in continual proportion A, B, C, D , and
 the extremes of them A, D be prime one to another,
 the last D shall not be to any other number E , as the
 first A is to the second B .

Suppose $A. B :: D. E$ then alternately $A. D ::$
 $B. E$. therefore seeing A and B are a the least in a 23. 7.
 their proportion, A b shall measure B , c and B b 21. 7.
 likewise C , and C the following number D . d and c 10. def. 7.
 so A shall measure the said number D . Wherefore d 11. ax. 7.
 A and D are not prime to one another; contrary
 to the Hypoth.

P R O P.

PROP. XVIII.

A, 4. B, 6. C, 9. Two numbers being given A, B, to consider if there may be a third number found proportional to them C.
Bq, 36.

a 8. ax. 7. If A measure Bq by any number C, a then AC
b 20. 7. = Bq. from whence b it is manifest that A. B :: B. C. Which was to be demonstrated.

But if A do not measure A, 6. B, 4. Bq, 16. Bq, there will not be any third proportional. For suppose A. B :: B. C a then AC = Bq. c and consequently $\frac{Bq}{A} = C$. namely A measures Bq. Which is against the Hypothesis.

PROP. XIX.

A, 8. B, 12. C, 18. D, 27. Three numbers being given A, B, C, to consider if a fourth proportional to them D may be found.
BC, 216.

a 9. ax. 7. If A measures BC by any number D, a then AD
b ax. 19. 7. = BC; b therefore it appears that A. B :: C. D. which was required.

But if A do not measure BC, then there can no fourth proportional be found; which may be shewn as in the prec. prop.

PROP. XX.

A, 2. B, 3. C, 5. More prime numbers may be given than any multitude whatsoever of prime numbers A, B, C, propounded.
D, 30. G----

a 38. 7. a Let D be the least which A, B, C, measure;
b 33. 7. If D + 1 be a prime, the case is plane; if composed, b then some prime number, conceive G, measures

asures $D + 1$; which is none of the three A, B, C ; For if it be, seeing it c measures the whole $D + 1$, d and the part taken away D , e it shall c *suppos.* also measure the remaining unit. *Which is absurd.* d *constr.* Therefore the propounded number of prime num- e *12. ax. 7.* bers is increased by $D + 1$, or at least by G .

P R P O. XXI.

⁵ A....⁵ E....³ B...³ F...² C..² G..² D 20.

If even numbers, how many soever, AB, BC, CD , be added together, the whole AD shall be even.

a Take $EB = \frac{1}{2} AB$, and $FC = \frac{1}{2} BC$, and GD a *6. def. 7.*
 $= \frac{1}{2} CD$. b it is plain that $EB + FC + GD = \frac{1}{2}$ b *12. 7.*
 AD . c therefore AD is an even number. *Which* c *6. def. 7.*
was to be demonstrated.

P R O P. XXII.

^I A^I F. B.....^I G. C....^I H.^I D..^I L. E 22.

⁹ If odd numbers, how many soever, AB, BC, CD, DE , ⁷ be added together, and their multitudes even, the ⁵ whole also AE shall be even. ³

A unit being taken from each odd number, there will a remain AF, BG, CH, DL , even num- a *7. def. 7.*
 bers, b and thence the number compounded of b *21. 9.*
 them will be even, add to them the c even number c *hyp.*
 made of the remaining units, and the d whole d *21. 9.*
 AE will thereby be even. *Which was to be dem.*

P R O P.

P R O P. XXIII.

$\begin{matrix} 7 & 5 & 1 \\ A & \dots & B & \dots & C & \dots & E & \dots & D & 15. \end{matrix}$
 If odd numbers how
 many soever AB , BC ,
 CD , be added together,
 and the multitude of
 them be odd, the whole AD shall be odd.

For CD one of the odd numbers being taken a-
 way, the number compounded of the others AC *a*
 is even. Whereto add CD — *b*, the whole AE
 is also even; wherefore the unit being restored
 the whole AD *c* will be odd. Which was to be dem.

P R O P. XXIV.

$\begin{matrix} 4 & 5 & 1 \\ A & \dots & B & \dots & D & \dots & C & 10. \end{matrix}$
 If an even number AB be
 taken away from an even
 number AC , that which re-
 mains BC shall be even.

For if BD ($BC - 1$) be odd, *a* BC ($BD + 1$)
 will be even. Which was to be dem. But if you say
 BD is even, because AC *b* is even, *c* thence AD
 will be so; *a* and consequently AC ($AD + 1$) will
 be odd, contrary to the Hypoth. therefore BC is even.
 Which was to be demonstrated.

P R O P. XXV.

$\begin{matrix} 6 & 1 & 3 \\ A & \dots & D & \dots & C & \dots & B & 10. \end{matrix}$
 If from an even number
 AB , an odd number AC be
 taken away, the remaining
 number CB shall be odd.

For $AC - 1$ (AD) *a* is even, *b* therefore DB is
 even; *c* and consequently CB ($DB - 1$) is odd.
 Which was to be demonstrated.

P R O P. XXVI.

$\begin{matrix} 4 & 6 & 1 \\ A & \dots & C & \dots & D & \dots & B & 11. \end{matrix}$
 If from an odd number
 AB be taken away an
 odd number CB , that
 which remaineth AC shall be even. For

For $AB - 1(AD)$ and $CB - 1(CD)$ a are even; a 7. def. 7.
 therefore $AD - CD (AC)$ is even. Which was to be demonstrated b 24. 9.

PROP. XXVII.

$A, D, \dots C, \dots B$ If from an odd number AB be taken away an even number CB , the residue AC shall be odd.

For $AB - 1(DB)$ a is even, and CB is supposed to be even; b therefore the residue CD is even; c therefore $CD + 1(CA)$ is odd. Which was to be demonstrated. a 7. def. 7. b 24. 9. c 7. def. 7.

PROP. XXVIII.

$A, 3$ If an odd number A multiplying an even number B produce a number AB , the number produced AB shall be even.

For AB a is compounded of the odd a hyp. and number A taken as many times as a unit is contained in B an even number. b Therefore AB is an even number. a 15 def. 7. b 21. 9.

Schol.

In like manner, if A be an even number, AB shall be an even number also.

PROP. XXIX.

$A, 3$ If an odd number A multiplying an odd number B , produce a number AB , the number produced AB shall be odd.

For AB a is compounded of the odd a 15 def. 7. number B taken as often as a unit is included in A likewise an odd number. b Therefore AB is an odd number. Which was to be demonstrated. b 23. 9.

Schol.

Schol.

$\frac{B, 12}{A, 3}$ (C, 4.

1. An odd number A measuring an even number B , measures the same by an even number C .

a 9. ax. 7. For if C be affirmed to be odd, then because a
b 29. 9. $B = AC$, b therefore B shall be odd; against the Hyp.

$\frac{B, 15}{A, 3}$ (C, 5

2. An odd number A measuring an odd number B , measures the same by an odd number C .

a 28. 9. For if C be said to be even, a then AC , or B will be even, contrary to the Hypoth.

$\frac{B, 15}{A, 3}$ (C, 5.

3. Every number (A and C) that measures an odd number B , is itself an odd number.

a 28. 9. For if either A or C be affirmed to be even, B shall be an even number, against the Hypoth.

P R O P. XXX.

$\frac{B, 24}{A, 3}$

(C, 8.

$\frac{D, 12}{A, 3}$

(E, 4

If an odd number A measures an even number B , it shall also measure the half of it D .

a hyp. a Let $\frac{B}{A}$ be $= C$. b then C is an even number.

b 1. schol. Therefore let E be $= \frac{1}{2} C$, then $Bc = CA$ $d = 2$
29. 9. EA $e = 2 D$. f therefore $EA = D$; g and conse-

c 9. ax. 7. quently $\frac{D}{A} = E$. Which was to be demonstrated.

d 1. 2.

e hyp.

f 7. ax. 1.

g 7. ax. 7.

P R O P. XXXI.

$A, 5. B, 8. C, 16. D ---$ If an odd number A be prime to any number B , it shall also be prime to the double thereof C .

a 3. schol. If it be possible, let some number D measure A and C , a then D measuring the odd number A shall be odd it self, b and so shall measure B the half of the even number C , therefore A and B are not prime one to another. Which is against the Hyp.
29. 9.
b 30. 9. Coroll.

Coroll.

It follows from hence, that an odd number which is prime to any number of double progression, is also prime to all the numbers of that progression.

P R O P. XXXII.

1. A, 2. B, 4. C 8, D, 16. *All numbers A, B, C, D, &c. in double progression from the binarie are evenly even only.*

It is evident that all these numbers 1, A, B, C, D, are even, and $b ::$, namely in a double proportion, c and so every less measures the greater by some one of them. d Wherefore all are evenly even. But for that A is a prime number, e no number beside these shall measure any of them. Therefore they are evenly even only. Which was to be dem.

a 6. def. 7.
b 20. def.
7.
c 12. 9.
d 8. def. 7.
e 13. 9.

P R O P. XXXIII.

A, 30. B, 15. *If of a number A, the half B be odd, the same A, is evenly odd only.*

Being an odd number B a measures A by two an even number, b therefore B is evenly odd. If you affirm it to be evenly even, c then some even number D measures it by an even number E. whence $2 B d = A d = D E$. e wherefore $2 : E :: D : B$. and therefore as 2 f measures the even number E, g so D an even number measures B an odd. Which is impossible.

a hyp.
b 9. def. 7.
c 8. def. 7.
d 9. ax. 7.
e 19. 7.
f 6. def. 7.
g 20. def. 7.

P R O P. XXXIV.

A, 24. *If an even number A be neither doubled from two, nor have it's half part odd, it is both evenly even and evenly odd.*

It is undoubtable, that A is evenly even, because the half of it is not odd. But because, if A be divided into two equal parts, and so continuing the bipartition

- a 7. def. 7. partition, we shall at length light upon some a odd number (not upon the number two, because A is not supposed to be doubled upward from two) which shall measure A by an even number. (for
 b 1. sch. 29. b otherwise A it self should be odd, against the
 9. Hyp.) Therefore A is evenly odd. Which was to be demonstrated.

P R O P XXXV.

A.....8.
 4.....8
 B...F.....G 12.
 C.....18.
 9.....6.....4.....8
 D.....H.....L.....K.....N 27.

If there be numbers in continual proportion. how many soever A, BG, C, DN, and the number FG be taken from the second, and KN from the last, equal to the first A; as the excess of the second BF is to the first A, so shall the excess of the last DK be to all the numbers that precede it, A, BG, C.

- a hyp. From DN take NL=BG, and NH=C. Because
 b 17. 5. DN. C (HN) a :: HN. BG (LN) a :: LN (BG.)
 c 12. 5. A (KN.) b therefore by dividing each, shall DH.
 d 3. ax. 1. HN :: HL. LN :: LK. KN. c wherefore DK. C+
 BG + A :: LK (d BF.) KN (A.) Which was to be demonstrated.

Coroll.

- e 18. 5. Hence e by compounding, DN+BG+C.A+BG
 +C :: BG. A.

P R O P XXXVI.

1. A, 2. B, 4. C, 8. D, 16.
 E, 31. G, 62. H, 124. L, 248. F, 496.
 M, 31. N, 465.
 P ---- Q ----

If from a unit be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole added together E be a prime number; and if this whole

whole E multiplying the last produce a number F, that which is produced F shall be a perfect number.

Take as many numbers E, G, H, L, likewise in double proportion continually; then a of equality A.D :: E. L. b therefore AL = DE c = F. d whence

$L = \frac{F}{2}$. Wherefore E, G, H, L, F, are :: in double

proportion. Let G - E be = M, and F - E = N ;

e then M. E :: N. E + G + H + L. f But M = E.

g therefore N = E + G + H + L. b therefore F = 1 +

B + C + D + E + G + H + L = E + N. Moreover

because D k measures DE (F) l therefore every one,

1, A, B, C, m measuring D, as m also E, G, H, L,

does measure F. And further, no other number

measures the said F. For if there do, let it be P,

which measures F by Q. n therefore PQ = F = DF.

o therefore E. Q :: P. D. therefore seeing A a prime

number measures D, p and so no other P measures

the same, q consequently E does not measure Q.

Wherefore E being supposed a prime number, r it

shall be prime to Q. s wherefore E and Q are the

least in their proportion ; t and so E measures P

as many times as Q does D, u therefore Q is one

of them A, B, C. Let it be B. seeing then of equa-

lity B. D :: E. H. x and so BH = DE = F = PQ. x

and so also Q. B :: H. P. y therefore H = P. there-

fore P is also one of them A, B, C, &c. against

the Hypothesis. Wherefore no other beside the

foresaid numbers measures E, and z consequently

F is a perfect number. Which was to be demon.

a 14. 7.

b 19. 7.

c hyp.

d 7. ax. 7.

e 35. 9.

f 3. ax. 1.

g 14. 5.

h 2. ax. 1.

k 7. ax. 7.

l 11. ax. 7.

m 11. 9.

n 9. ax. 7.

o 19. 7.

p 13. 9.

q 20. def. 7.

r 31. 7.

s 23. 7.

t 21. 7.

u 13. 7.

x 19. 7.

y 14. 5.

z 22. def. 7.

The End of the ninth Book.

N

T H E

The TENTH BOOK
OF
E U C L I D E ' S
E L E M E N T S.

Definitions.

- I. **C**ommenfurable magnitudes are thofe, which are meafured by one and the fame meafure.



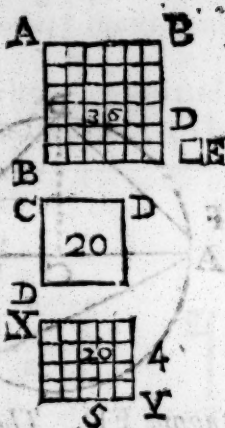
The note of commenfurability is $\sqsubset$, as $A \sqsubset B$; that is, the line A of 8 foot is commenfurable to the line B of 13 foot; becaufe D a line of one foot meafures both A and B. Alfo $\sqrt{18} \sqsubset \sqrt{50}$; becaufe $\sqrt{2}$ meafures both $\sqrt{18}$ and $\sqrt{50}$. For $\sqrt{1^2} = \sqrt{9} = 3$. and $\sqrt{5^2} = \sqrt{25} = 5$. wherefore $\sqrt{18} : \sqrt{50} :: 3 : 5$.

- II. Incommenfurable magnitudes are fuch, of which no common meafure can be found.

Incommenfurability is denoted by this mark $\sqsupset$; as $\sqrt{6} \sqsupset \sqrt{25}$ (5;) that is, $\sqrt{6}$ is incommenfurable to the number 5, or to a magnitude defigned by that number; becaufe there is no common meafure of them, as fhall appear hereafter.

- III. Right lines are commenfurable in power, when the fame fpace does meafure their fquares.

The mark of this commensurability is $\sqcup$; as $AB \sqcup CD$. i. e. the line AB of 6 foot is in power commensurable to the line CD , which is expressed by $\sqrt{20}$. because E the space of one foot square does as well measure ABq (36) as the rectangle XY (20) to which the square of the line CD ($\sqrt{20}$) is equal. The same note $\sqcup$ sometimes signifies commensurable in power only.



IV. Lines incommensurable in power are such, to whose squares no space can be found to be a common measure.

This incommensurability is denoted thus; $5 \sqcup \sqrt{8}$. i. e. the numbers or lines 5, and $\sqrt{8}$ are incommensurable in power, because their squares 25 and 8 are incommensurable.

V. From which it is manifest, that to any right line given right lines infinite in multitude are both commensurable and incommensurable; some in length and power, others in power only. The right line given is called a Rational line.

The note of which is ρ .

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rational, ρ .

VII. But such as are incommensurable to it, are called Irrational,

And denoted thus $\rho\alpha$.

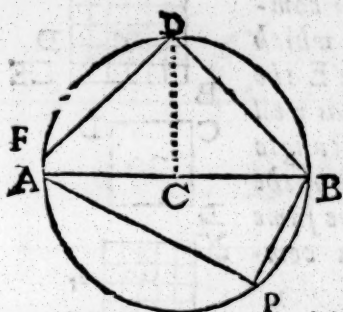
VIII. Also the square which is made of the said given right line is called Rational, $\rho\nu$.

IX. And likewise such figures as are commensurable to it, are Rational $\rho\alpha$.

X. But such as are incommensurable, Irrational $\rho\alpha$.

XI. And those right lines also, which contain them in power, are Irrational ρ .

Schol.



That the last seven definitions may be render'd more clear by an example, let there be a circle ADBP, whose semidiameter is CB, inscribe therein the sides of the ordinate figures, as of a Hexagone BP, of a Triangle AP, of a Square BD, of a Pentagon FD. Therefore, if according to the 5. def. the semidiameter CB be the Rational line given, expressed by the number 2, to which the other lines BP, AP, BD, FD, are to be compared, then $BP = BC = 2$. wherefore BP is $\dot{p} \sqcap BC$, according to the 6. def. Also AP $b = \sqrt{12}$ (for ABq (16) — BPq (4) = 12) therefore AB is $\dot{p} \sqsubset BC$ likewise according to the 6. def. and APq (12) is $\dot{p} \nu$ by the 9. def. Moreover BD $b = \sqrt{DCq + BCq} = \sqrt{8}$; whence BD is $\dot{p} \sqcup BC$; and BDq $\dot{p} \nu$. Lastly, FDq = 10 — $\sqrt{20}$ (as shall appear by the praxis to be delivered at the 10. 13.) shall be $\dot{p} \nu$, according to the 10. def. and consequently FD = $\sqrt{10 - \sqrt{20}}$ is $\dot{p}$, according to the 11. def.

A Postulate.

That any magnitude may be so often multiplied, till it exceed any magnitude whatsoever of the same kind.

Axiomes.

1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.

2. A

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue

P R O P. I.

Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more than half (AH) and from the residue (HB) be again taken away more than half (HI) and this be done continually, there shall at length be left a certain magnitude IB, less than the less of the magnitudes first given C.

a Take C so often, till its multiplex do somewhat exceed AB, and there be $DF = FG = GE = C$. Take from AB more than half, HA, and from the remainder HB more than half, viz. HI, and so continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE, which is not less than $\frac{1}{2}$ DE, is greater than HB, which is less than $\frac{1}{2}$ AB $\supset$ DE. And in like manner GE, which is not less than $\frac{1}{2}$ FE, is greater than IB $\supset$ HB. therefore C, or $GE \supset IB$. Which was to be dem.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.



a post. 10.

P R O P. II.



Two unequal magnitudes being given (AB, CD) if the less AB be continually taken from the greater CD, by an interchangeable subtraction, and the residue do not measure the magnitude going before, then are the magnitudes given incommensurable.

If it be possible, let some magnitude E be the common measure. Then because AB taken from CD, as often as it can be, leaves a magnitude FD less than it self, and FD taken from AB leaves GB, and so forward a therefore at length some magnitude GB shall be left. therefore E *b* measuring AB, *c* and so CF, *b* and the whole CD, *d* shall also measure the residue FD. *c* consequently also AG; *d* wherefore it shall likewise measure the remainder GB, less than it self. Which is absurd.

a 1. 10.

b hyp.

c 2. ax. 10.

d 3. ax. 10.

P R O P. III.



Two commensurable magnitudes being given AB, CD, to find out their greatest common measure FB.

Take AB from CD, and the residue ED from AB, and FB from ED, till FB measure ED (which will come to pass at length, *a* because by the Hyp. AB $\nmid$ CD) FB shall be the magnitude required.

For FB *b* measures ED, *c* and so also AF; but it measures it self too, *d* therefore likewise AB, *c* and consequently CE, *d* and so the whole CD. Wherefore FB

is the common measure of AB, CD. If you affirm G to be a common measure greater than that, then G measuring AB and CD, *e* measures also CE and *f* the remainder ED, *e* and so AF; and *f* consequently the remainder FB, the greater the less. Which is absurd.

a 2. 10.

b constr.

c 2. ax. 10.

d 1. ax. 10.

e 2. ax. 10.

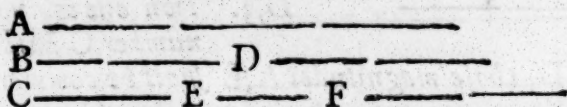
f 3. ax. 10.

Coroll.

Coroll.

Hence, A magnitude that measures two magnitudes, does also measure their greatest common measure.

P R O P. IV.



Three commensurable magnitudes being given A, B, C, to find out their greatest common measure.

a Find out D the greatest common measure of any two A, B; a also E the greatest common measure of D and C. therefore E is the magnitude sought for. a 3. 10.

a For it is clear, E measuring D and C, b does measure the three, A, B, C. Conceive another magnitude F greater than that to measure them; b const. E 2. ax. 10. c then F measures D, c and consequently E the greatest common measure of D, and C, the greater the less. Which is absurd. c cor. 3. 10.

Coroll.

Hence also it appears, that if a magnitude measure three magnitudes, it shall likewise measure their greatest common measure.

P R O P. V.

Commensurable mag- A ————— D. 4.
nitudes A, B, have such C ————— F. 1.
proportion one to ano- B ————— E. 3.
ther, as number hath to number.

a C being found the greatest common measure of A, B; as often as C is contained in A and B, so often is 1 contained in the numbers D and E; a 3. 10. b therefore C. A :: 1. D; wherefore inversely A. C :: D. 1. b but likewise C. B :: 1. E. c therefore of equality A. B :: D. E :: N. N. Note, The letter N only signifies number in general, and refers not to any particular space or magnitude as the other letters do, and is to be read, as A to B, so D to E, and so number to number. Which was to be dem. b 17. def. 7

The tenth Book of

P R O P. VI.

E ————— F . . . If two magnitudes
 A ————— C , 4. A, B , have such propor-
 B ————— D , 3. tion one to another, as
 number C hath to num-
 ber D , those magnitudes A, B shall be commensurable,

a f. b. 10. 6. What part 1 is of the number C , a that let E
 b constr. be of A . Therefore because $E.A :: 1.C$ and $A.B ::$
 c hyp. $c :: C.D$. d therefore of equality shall $E.B :: 1.D$.
 d 21. 5. Wherefore seeing 1 measures the number D , f
 e 5. ax. 7. likewise E measures B ; but it g also measures A .
 f 20. def. 7. b therefore $A \sqsubset B$. Which was to be demonstrated.
 g constr.
 h 1. def. 10.

P R O P. VII.

A ————— Incommensurable magnitudes
 B ————— A, B , have not that proportion
 one to another, which number
 hath to number.

a 6. 10. If you affirm $A.B :: N.N$. a then $A \sqsubset B$. against
 the Hypothesis.

P R O P. VIII.

A ————— If two magnitudes A, B ,
 B ————— have not that proportion one
 to another, which number hath
 to number, those magnitudes are incommensurable.

e 5. 10. Conceive $A \sqsubset B$. a then $A.B :: N.N$. con-
 trary to the Hypothesis.

P R O P. IX.

A ————— The squares described of right lines
 B ————— commensurable in length, have that
 E , 4. proportion one to another, that a
 F , 3. square number hath to a square num-
 ber. And squares, which have that
 proportion one to another, that a square number hath

a square number shall also have their sides commensurable in length. But such squares as are made of right lines incommensurable in length, have not that proportion one to another, which a square number hath to a square number. And squares which have not such proportion one to another as a square number hath to a square number, have not their sides commensurable in length.

1. Hyp. $A \sqsupset B$. I say $Aq.Bq :: Q.Q$.

For a let $A.B ::$ number E. number F. therefore a 5. 10.

$Aq \frac{A}{B}$ (b $\frac{A}{B}$ twice) $c = \frac{E}{F}$ twice. $d = \frac{Eq}{Fq}$. e therefore b 20. 6. c sch. 23. 5.

$Aq.Bq :: Eq.Fq :: Q.Q$. Which was to be dem. d 11. 8.

2. Hyp. $Aq.Bq :: Eq.Fq :: Q.Q$. I say $A \sqsupset B$. For e 11. 5.

$\frac{A}{B}$ twice (f $\frac{Aq}{Bq}$ $g = \frac{Eq}{Fq}$ $h = \frac{E}{F}$ twice. i therefore f 20. 6. g hyp.

$A.B :: E.F :: N.N.$ k wherefore $A \sqsupset B$. Which h 11. 8.

was to be demonstrated. i sch. 23. 5.

3. Hyp. $A \sqsupset B$. I deny that $Aq.Bq :: Q.Q$. For k 6. 10.

suppose $Aq.Bq :: Q.Q$. then $A \sqsupset B$, as is shewn before, against the Hypothesis.

4. Hyp. Not $Aq.Bq :: Q.Q$. I say that $A \sqsupset B$.

For conceive $A \sqsupset B$. then $Aq.Bq :: Q.Q$ as above, against the Hypothesis.

Coroll.

Lines $\sqsupset$ are also $\sqsupset$ but not on the contrary. And lines $\sqsupset$ are not the efore $\sqsupset$. but $\sqsupset$ are also $\sqsupset$.

PROP.

P R O P. X.



a 5. 10.
b 6. 10.
c 7. 10.
d 8. 10.

C A B D

If four magnitudes be proportional (C. A :: B. D) and the first C be commensurable to the second A, the third B shall be commensurable to the fourth D. And if the first C be incommensurable to the second A, also the third B shall be incommensurable to the fourth D.

If $C \sqsubset A$, a then $C. A :: N. N b :: B. D. b$ therefore $B \sqsubset D$. But if $\sqsupset A$, c then shall not $C. A :: N. N :: B. D. e$ wherefore $B \sqsupset D$. Which was to be demonstrated.

Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like will satisfy this Lemma, as those numbers which have superparticular, superbipartient, or double proportion; or any two prime numbers, See schol. 27. 8.

Lemma 2.

B. 5. K — 1 — 1 — 1 — 1 — M
C. 3. H — 1 — 1 — R

a sch. 10. 6. To find out a line HR, to which a right line given KM hath the proportion of two numbers given B, C.
b 3. 1. a Divide KM into as many equal parts as there are units in the number B, and let as many of these, as there are units in the number C, b make the right line HR. it is manifest that $KM, HR :: B. C$.

Lemma 3.

To find out a line D, to the square of which the square of a right line given KM hath the proportion of two numbers given B, C.

a 2. lem. 10. Allow $B. C a :: KM. HR$. and between KM and HR, b find a mean proportional D. Therefore $KM q D q c :: KM. HR d :: B. C$.
10.
b 13. 6.
c 20 6.
d constr.

P R O P.

PROP. XI.

To find two right lines in-
commensurable to a right
line given A, one D in
length only, the other E in
power also.



1. Take the numbers B, C, a so that there be
not B. C :: Q. Q. b let B. C :: Aq. Dq. c it is plain
that A $\square$ D. But Aq d $\square$ Dq. Which was to be
done.

a 1.lem.10.
10.
b 3.lem.10.
10.

2. d Make A. E :: E. D. I say Aq $\square$ Eq. For A.
D e :: Aq. Eq. therefore since A $\square$ D, as before ;
f therefore Aq $\square$ Eq. Which was to be done.

c 9. 10.
d 6. 10.
d 13. 6.
e 20. 6.
f 10. 10.

PROP. XII.

Magnitudes (A,B) commensurable to the
same magnitude C, are also commensurable
one to the other.

Because A $\square$ C, and C $\square$ B, a let A.
C :: N. N :: D. E, and C. D, 8. E, 8.
B :: N. N :: E. G. b take F, 2. G, 3.
three numbers H, I, K, the H, 5 I, 4. K, 6.



a 5. 10.
b 4. 8.

least :: in the proportions of D to E, A B C
and F to G. Now because A. C c :: D. E c constr.
e :: H. I. and C. B e :: F. G :: I. K. d therefore of d 22. 5.
equality A. B :: H. K :: N. N. e therefore A $\square$ B. e 6. 10.
Which was to be demonstrated.

Schol.

Hence, Every right line commensurable to a
rational line is also it self rational. And all right
lines rational are commensurable to one another,
at least in power. Also, every space commen-
surable to a rational space is rational too : And all
rational spaces are commensurable one to another.
For magnitudes whereof one is rational, the other
rational, are incommensurable amongst them-
selves.

12. 10. and
def. 6.
def. 9.
def. 7. and
10.

PROP.

P R O P. XIII.

A ————
C ————
B ————

If there be two magnitudes A, B, and one of them A commensurable to a third C, but the other B incommensurable, those magnitudes A, B are incommensurable.

a hyp.

b 12. 10.

Conceive $B \sqsubset A$. then being $C a \sqsubset A$, b therefore $C \sqsubset B$, against the hypothesis.

P R O P. XIV.

|||
A B C

If there be two magnitudes commensurable A, B; and one of them A incommensurable to any other magnitude C, the other also B shall be incommensurable to the same C.

a hyp.

b 12. 10.

Imagine $B \sqsubset C$. then for that $A a \sqsubset B$, b therefore $A \sqsubset C$, against the hyp.

P R O P. XV.

A ————
B ————
C ————
D ————

If four right lines be proportional (A. B :: C. D) and the first A be in power more than the second B by the square of a right line commensurable to it self in length, then also the third C shall be more in power than the fourth D by the square of a right line commensurable to it self in length. But if the first A be more in power than the second B by the square of a right line incommensurable to it self in length, then shall the third C be more in power than the fourth D by the square of a right line incommensurable to it self in length.

a hyp.

b 21. 6.

c 17. 5.

d 22. 6.

e cor 4. 5.

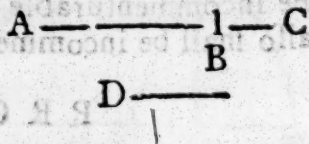
f 22. 5.

For because A. B a :: C. D. b therefore $Aq. Bq. Cq. Dq.$ c therefore by division $Aq - Bq. Bq :: Cq - Dq. Dq.$ d wherefore $\sqrt{Aq - Bq. B} :: \sqrt{Cq - Dq. D}$ e and so inversely $B \sqrt{Aq - B} :: D \sqrt{Cq - D}$ f therefore of equality $A. \sqrt{Aq - Bq} :: C. \sqrt{Cq - Dq}$ consequently if $A \sqsubset B$, o

$\square \sqrt{Aq - Bq}$ g then likewise $C \square$, or $\square \sqrt{}$ g 10. 10.
 $Cq - Dq$. Which was to be demonstrated.

P R O P. XVI.

If two magnitudes commensurable AB, BC, be composed, the whole magnitude AC shall be commensurable to each of the parts AB, BC. And if the whole magnitude AC be commensurable to either of the parts AB, or BC, those two magnitudes given at first AB, BC, shall be commensurable.



1. Hyp. a Let D be the common measure of AB, a 3. 10.
 BC; b 10 D measures AC. and therefore AC $\square$ b 1. ax. 10.
 AB, and BC. Which was to be demonstrated. c 1. def. 10.

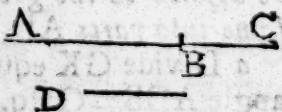
2. Hyp. a Let D be the common measure of AC,
 AP. d therefore D measures AC - AB (BC) and d 3. ax. 10.
 consequently AB $\square$ BC. Which was to be dem.

Coroll.

Hence it follows, if a whole magnitude composed of two be commensurable to any one of them, the same shall be commensurable to the other also.

P R O P. XVII.

If two incommensurable magnitudes AB, BC, be composed, the whole magnitude also AC shall be incommensurable to either of the two parts AB, BC. And if the whole magnitude AC be incommensurable to one of them AB, the magnitudes first given AB, BC, shall be incommensurable.



1. Hyp. 11 it can be, let D be the common measure of AC, AB. a therefore D measures AC - a 3. ax. 10.
 AB (BC) b and therefore also AB $\square$ BC, against b 1. def. 10.
 the hypothesis.

2. Hyp.

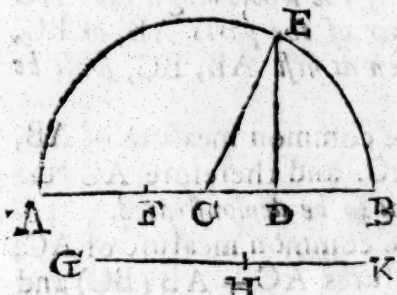
c 16. 10.

2. Hyp. Conceive $AB \perp BC$. & therefore $AC \perp AB$. against the hypothesis.

Coroll.

Hence also, If one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

P R O P. XVIII.



If there be two unequal right lines AB, GK, and upon the greater AB a parallelogram ADB equal to the fourth part of a square made of the less line GK, and wanting in figure by a square

be applied, and divide the said AB into parts commensurable in length AD, DB; then shall the greater line AB be more in power than the less GK by the square of a right line FD commensurable in length to the greater. And if the greater B be in power more than the less GK by the square of the right line FD commensurable unto it self in length, and a parallelogram ADB equal to the fourth part of the square made of the less line GK, and wanting in figure by a square be applied to the greater AB, then shall it divide the same into parts AD, DB commensurable in length.

a 10. 1.

b 28. 6.

c 8. 2.

d constr. &

4. 2.

e 16. 10.

g cor. 16. 10

h cor. 16. 10

k 12. 10.

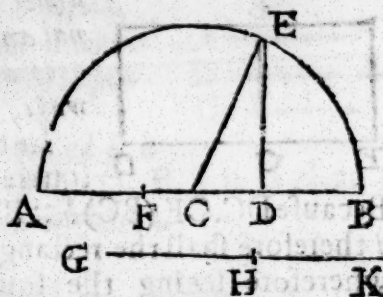
l 16. 10.

a Divide GK equally in H, and b make the rectangle ADB = GHq. Cut off AF = DB. then is AB c = 4 ADB d (4 GHq or GKq) + FDq. Now in the first place, if $AD \perp DB$, then shall AB e $\perp$ BD e $\perp$ 2 DB f (AF + DB, or AB - FD) b therefore AB $\perp$ FD. Which was to be dem. But secondly if AB $\perp$ FD, b then shall AB $\perp$ AB - FD (2 DB k therefore AB $\perp$ DB. l wherefore AD $\perp$ DB. Which was to be demonstrated.

P R O P.

PROP. XIX.

If there be two right lines unequal AB, GK, and to the greater AB be applied a parallelogram ADB equal to the fourth part of a square made upon the less GK. and wanting in figure by a square, and also thus



applied divide the said AB into parts AD, DB incommensurable in length; the greater line AB shall be in power more than the less GK by the square of the right line FD incommensurable to the greater in length. And if the greater line AB be more in power than the less GK by the square of a right line FD incommensurable unto it self in length, and if also upon the greater AB be applied a parallelogram ADB equal to the fourth part of the square of the less GK and wanting in figure by a square, then shall it divide the said greater line AB into parts incommensurable in length AD, DB.

Suppose all the same that was done and said in the prec. prop. Therefore first, If $AD \perp DB$, then shall $AB \perp DB$. *b* Wherefore $AB \perp 2 DB$ *a* $(AB - FD)$ *c* therefore $AB \perp FD$. Which was to be demonstrated.

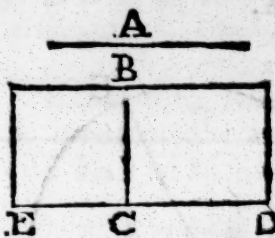
a 17. 10.
b 13. 10.
c cor. 17.

Secondly, If $AB \perp FD$, then $AB \perp AB - FD$ ($2 DB$;) *d* wherefore $AB \perp DB$, *e* and consequently $AD \perp DB$. Which was to be demonstrated.

10.
d 13. 10.
e 17. 10.

PROP.

P R O P. XX.



a 46. 1.

b 1. 6.

c hyp.

d 10. 10.

e hyp. and

9. def. 10.

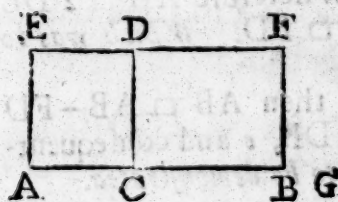
i 12. 12.

Because $DC.CE$ (BC) $b :: BD.BE$. and DC $c \sqsubset BC$,
 d therefore shall the rectangle BD be $\sqsubset$ square BE .
 wherefore seeing the square BE $e \sqsubset Aq$, shall
 also BD be $\sqsubset Aq$. and so the rectangle BD is $\dot{p}$.
Which was to be demonstrated.

Note, There are three kinds of lines rational commensurable one to another. For either of two lines rational commensurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it, notwithstanding both of them are commensurable to it in length; or lastly both of them are commensurable to the rational line given only in power. And these are the ways which the present theorem speaks of.

In numbers, let there be $BC, \sqrt{8} (2\sqrt{2})$ and $CD \sqrt{18} (3\sqrt{2})$ then shall the rectangle $BD = \sqrt{144} = 12$.

P R O P. XXI.



If a rational rectangle DB be applied to a rational line DC , it makes the breadth thereof CB rational and commensurable in length to that line DC whereunto DB is applied.

a 2. 6.

b hyp.

c sch. 12. 10.

d 20. 10.

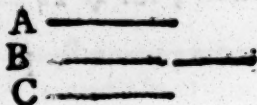
e sch. 12. 10.

Let G be propounded $\dot{p}$, and the square D described on EC . because $BD.DA :: a BC.CA$ and $BD.DA$ b are $\dot{p} \propto c$ and so $\sqsubset$. d therefore B $\sqsubset CA$. but CD (CA) is $\dot{p}$. e therefore BC is $\dot{p}$.
Which was to be demonstrated.

In numbers, let there be the rectangle DB, 12, and DC, $\sqrt{8}$. then shall CB, $\sqrt{18}$. but $\sqrt{18} = 3\sqrt{2}$. and $\sqrt{8} = 2 \times \sqrt{2}$.

Lemma.

To find out two right lines rational commensurable only in power.

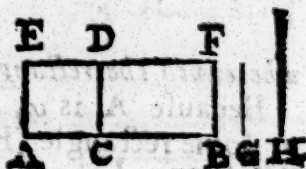


Let A be propounded ρ . a
Take B $\perp$ A, a and C $\perp$ B. b it is clear that B and C are the lines required.

a 11. 10.
b fch. 12.
10.

P R O P. XXII.

A rectangle DB comprehended under right lines rational DC, CB commensurable in power only, is irrational: and the right line H, which containeth that rectangle in power is irrational, and called a Medial line.



Let G be the propounded ρ , and the square DA described on DC, and let Hq = DB. Because AC. CB a :: DA. DB. b and AC $\perp$ CB, c shall be DA $\perp$ DB (Hq.) d but Gq $\perp$ DA. e therefore Hq $\perp$ Gq. f wherefore H is ρ . Which was to be dem. and let it be called a Medial line, because AC. H :: H. CB.

a 2. 6.
b hyp.
c 10. 10.
d hyp. and
9. def. 10.
e 13. 10.
f 11. 10.

In numbers, let there be DC, 3. and CB, $\sqrt{6}$. then shall the rectangle be DB (Hq) $\sqrt{54}$. wherefore H is $\sqrt{54}$.

The note of a medial line is μ , of a medial rectangle $\mu\nu$, of more together $\mu\alpha$.

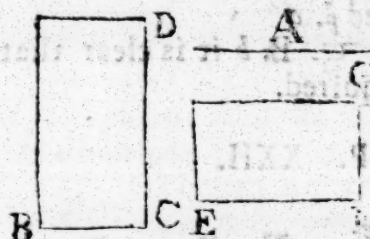
Schol.

Every rectangle that can be contained under two right lines rational commensurable only in power, is medial, although it be contained under two right lines irrational: and every medial rectangle may be contained under two right lines rational, commensurable only in power; as for example,

O

example, the $\sqrt{24}$ is $\mu\nu$, because it is contained under $\sqrt{3}$, and $\sqrt{8}$, which are ρ , τ . although it may be contained under $\nu\sqrt{6}$, and $\nu\sqrt{96}$ irrationals; for $\sqrt{24} = \nu\sqrt{576} = \nu\sqrt{6} \times \nu\sqrt{96}$.

P R O P. XXIII.

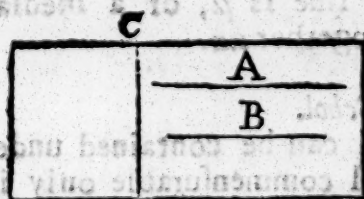


If the rectangle BD made of a medial line A, be applied on a rational line BC, it makes the breadth CD rational, and incommensurable in length to the line BC,

whereunto the rectangle BD is applied.

- a sch. 22. Because A is μ , a therefore shall Aq be equal
 10. to some rectangle (FG) contained under EF and
 b 1. ax. 1. $FG \rho \tau$. b therefore $BD = EG$. d whence BC.
 c 14. 6. $EF :: FG.CD$. d therefore $BCq.EFq :: FGq.CDq$.
 d 22. 6. But BCq and EFq e are $\rho a, f$ and so τ . g therefore
 e hyp. $FGq \tau CDq$. Wherefore being FG is ρ, h there-
 f sch. 12. fore CD shall be ρ . Moreover, because $EF.FG \rho$
 10. $:: EFq.EG (BD;)$ for that $EF \tau FG$, e shall
 g 6. 10. EFq be τBD . But $EFq m \tau CDq$. h there-
 h sch. 12. fore the rectangle BD τCDq . Whence being
 10. CDq . $BD o :: CD.BC$. p shall CD be τBC .
 k 1. 6. therefore, &c.

P R O P. XXIV.



A right line B commensurable to a medial line A, is also a medial line.

Upon CD ρa make the rectangle $CE = Aq$; a and the rectangle $CF = Bq$. Ee-

- a 11. 6. cause Aq (CE) is $\mu\nu$, b and CD ρ, c therefore shall
 b hyp. the latitude DE be $\rho \tau CD$. But for that CE. CF
 c 23. 10. d ::

d :: ED. DF. and CE $\perp$ CF, *f* therefore ED $\perp$ DF. *g* therefore DF is ρ $\perp$ CD. *h* whence the rectangle CF (Bq) is $\mu\nu$, and so B is μ . Which was to be demonstrated.

Obf. that the note $\perp$ for the most part signifies commensurable in power only, as in this and the precedent demonstrations, &c.

Coroll.

Hereby it is manifest that a space commensurable to a medial space, is also medial.

Lemma.

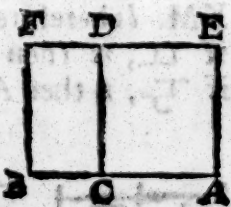
To find out the right lines medial A, B, commensurable in length, and also two, A, C, commensurable only in power.

a Let A be any μ , *b* take B $\perp$ A, and *c* C $\perp$ A *d* it appears to be done.

PROP. XXV.

A rectangle DB contained under DC, CB medial right lines commensurable in length, is medial.

Upon DC describe the square DA. Being AC (DC.) CB *a* :: DA. DB, and DC $\perp$ CB; *b* shall DA $\perp$ DB. *c* therefore DB is $\mu\nu$. Which was to be demonstrated.

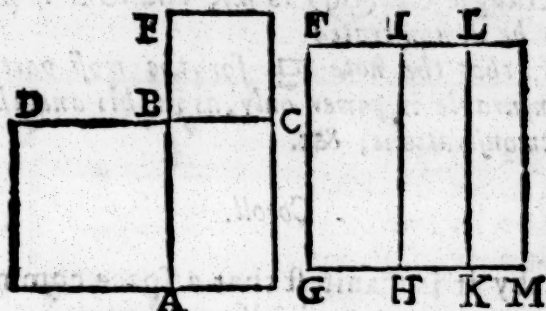


a lem. 22.
10. and 13.
6.
b 2. lem.
10. 10.
c 3. lem.
10. 10.
d constr.
and 24. 10.
a 1. 6.
b 10. 10.
c 24. 10.

O 2

PROP.

PROP XXVI.



A rectangle AC comprehended under medial right lines AB, BC commensurable only in power, is either rational or medial.

a 46. 1.
b cor. 16.
6.
c hyp. &
24. 10.
d 23. 10.
e 10. 16.
f 20. 10.
g sch. 22. 6.
h 1. 6.
k 17. 6.
l 12. 16.
m 20. 10.
n 22. 10.

Upon the lines AB, BC, *a* describe the squares AD, CE; and upon FG *b* make the rectangles FH, = AD, *b* and IK = AC. *b* and LM = CE. The squares AD, CE, that is, the rectangles FH, LM, *c* are *ua* and $\square$. therefore GH, KM, having the same proportion *d* are β , *e* and $\square$ *f* therefore GH $\times$ KM is $\beta\gamma$. But because AD, AC, CE, that is, FH, IK, LM, *g* are $\div$; *b* and so GH, HK, KM also $\div$; *k* thence HK $\times$ GH = GH $\times$ KM. I therefore HK is β , or $\square$, or γ IH (GF;) if $\square$, *m* then the rectangle IK or AC is $\beta\gamma$. but if γ , *n* then AC is $\mu\nu$. Which was to be dem.

Lemmd.



If A and E be γ only, Then first, shall Aq, Eq, Aq

a hyp. and
16. 10.
b 1. 6.
c hyp.
d 10. 10.
e 14. 10.
f 14. 10.

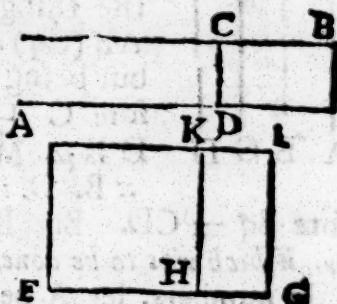
+ Eq, Aq - Eq *a* $\square$. And secondly Aq, Eq, Aq + Eq, Aq - Eq $\square$ AE and 2 E. For A.E *b* :: Aq. AE *b* :: AE. Eq. therefore seeing A *c* $\square$ E. *d* shall Aq $\square$ AE, *e* and 2 AE. also Eq *d* $\square$ AE, *e* and 2 AE. wherefore because Aq + Eq $\square$ Aq and Eq; and Aq - Eq $\square$ Aq and Eq. *f* therefore shall Aq + Eq, *f* and Aq - Eq be $\square$ AE, and 2 AE.

Hence

Hence also thirdly, $Aq, Eq, Aq + Eq, Aq - Eq, 2 AE$ $g \sqcup Aq + Eq + 2 AE$; and $Aq - Eq$ $g \text{ 14. 1. } - 2 AE. g$ and $Aq + Eq + 2 AE \sqcup Aq + Eq - 2 AE. h (Q. A - E.)$ and 17.10. hcor. 7.12.

PROP XXVII.

A medial rectangle AB exceedeth not a medial rectangle AC by a rational rectangle DB.



a cor. 16.6.

Upon EF ρ , a make $EG = AB$, a and $EH = AC$. The rectangles AB, AC, i. e. EG, EH, b are $\mu\alpha$; c therefore FG and FH are $\rho \sqcup EF$. Whence, if KG, d i. e. DB be $\rho\gamma$, e then shall HG be $\sqcup HK$; f wherefore HG $\sqcup FH$. g and consequently $FGq \sqcup FHq$. But FH is ρ . h therefore is FG ρ . but FG was ρ before. Which is contradictory.

b hyp.

c 23. 10.

d 3. ax. 1.

e 21. 10.

f 13. 10.

g lem. 16.

10.

h sch. 12.

10.

Schol.

1. A rational rectangle AE exceeds a rational rectangle AD by a rational rectangle CE.



For AE a $\sqcup AD$, b therefore AE $\sqcup CE$. c wherefore CE is $\rho\gamma$. Which was to be dem.

a hyp.

b cor. 16.

10.

2. A rational rectangle AD joined with a rational rectangle CF makes a rational rectangle AF.



c sch. 12.

10.

For AD a $\sqcup CF$. b wherefore AF $\sqcup AD$ and CF; c and so AF is $\rho\gamma$. Which was to be dem.

a sch. 12.

10.

b 16. 11.

c sch. 12.

10.

O 3

PROP.

P R O P. XXVIII.

a lem. 21.

10.

b 13. 6.

c 12. 6.

d 22. 10.

e constr.

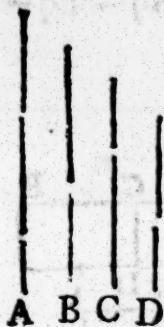
f 10. 10.

g 14. 10.

h 17. 6.

h sch. 12.

10.



A B C D

fore Bq = CD.

But Bq is $\rho\gamma$.

Which was to be done.

In numbers, let A be $\sqrt{2}$;and B $\sqrt{6}$.therefore C is $v\sqrt{12}$.make $\sqrt{2}$. $\sqrt{6} :: v\sqrt{12}$.

D. or

 $v\sqrt{4} \cdot v\sqrt{36} :: v\sqrt{12}$.D, then shall D be $v\sqrt{108}$.but $v\sqrt{12} \times v\sqrt{108} = v\sqrt{1296} = \sqrt{36}$ $= 6$.therefore CD is 6, likewise C, D :: 1. $\sqrt{3}$.wherefore C $\sqsupset$ D.

To find out medial lines (C and D.) which contain a rational rectangle CD.

a Take A and B $\rho\gamma$. b make A C :: C. B. c and A. B :: C. D. I say the thing required is done. For AB (Cq) d is $\mu\nu$, d whence C is μ . but being that A. B :: C. D. f therefore C $\sqsupset$ D. g and consequently D is μ . Moreover by inversion A. C :: B. D. i. e. C. C :: B. D. b therefore Bq = CD. But Bq is $\rho\gamma$. h therefore CD is $\rho\gamma$. Which was to be done.

P R O P. XXIX.

a lem, 21.

10.

b 13. 6.

c 12. 6.

d 17. 6.

e 22. 20.

f constr.

g 10. 10.

h 24. 10.

k constr.

and cor. 45.

l 16 6.

m 22. 6.



A D B C E

g whence D $\sqsupset$ E.therefore h E is μ .

Moreover B. C f :: D. E. and by inversion B. D :: C. E. i. e.

D. A :: C. E. l therefore D. E = AC. But AC m

is $\mu\nu$. therefore DE is $\mu\nu$. Which was to be done.In numbers, let A be 20. and B, $\sqrt{200}$, and C, $\sqrt{80}$. Therefore D is $\sqrt{\sqrt{80000}}$; and E $v\sqrt{12800}$.Therefore DE = $\sqrt{\sqrt{1024000000}} = 32000$.and D. E :: $\sqrt{10}$. 2. wherefore D $\sqsupset$ E.

To find out medial right lines commensurable in power only, D and E, containing a medial rectangle DE.

a Take A, B, C $\rho\gamma$. make A. D b :: D. B. c and B. C :: D. E. I say the thing desired is performed.

For AB d = Dq. and AB e is $\mu\nu$, therefore D is μ ; and B f $\sqsupset$ C, g whence D $\sqsupset$ E. therefore h E is μ . Moreover B. C f :: D. E. and by inversion B. D :: C. E. i. e. D. A :: C. E. l therefore D. E = AC. But AC m

is $\mu\nu$. therefore DE is $\mu\nu$. Which was to be done.

Schol.

Schol.

To find out two plane numbers, like or unlike.

A, 6. C, 12

B, 4. D, 8.

Take any four numbers proportional A. B :: C. D.

AB, 24. CD, 96.

it is manifest that AB and CD are like plane numbers.

A, 6. C, 5.

B, 4. D, 8.

And you may find out as many unlike plane numbers, as you please, by help of *Schol.* 27. 8.

AB, 24. CD, 40.

Lemma.



To find out two square numbers (DEq and CDq) so that the number composed of them (CEq) be square also.

Take AD, DB like plane numbers (of which let both be equal, or both odd) viz. AD, 24. and DB, 6. The total of these (AB) is 30; the difference (FD) 18. half of which (CD) is 9. *a* 18. 8. Now the like plane numbers AD, DB, have one mean number proportional, namely DE. therefore it is evident that every of those numbers CE, CD, DE, are rational, and by consequence CEq (*b* $CDq + DEq$) *b* 47. 1. is the square number required.

Whereby it will be easy to find out two square numbers, the excess of which is a square or not a square number, namely by the same construction shall $CEq - CDq = DEq$.

But if AD, DB be plane numbers unlike, the

c 3. *ax* 1.

medial proportional line(DE) shall not be a rational number, and so neither shall the excess (DEq) of the square numbers, CEq, CDq. be a square number.

Lemma 2.

2 To find out two such square numbers B, C, as the number compounded of them D is not square. Also to divide a square number A into two numbers B, C, not squares.

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be $\equiv 4B$, and $D = B + C$. I say the matter is done.

For B is Q. by the constr. likewise because B.C
 a 24. 8. $\therefore 1.4 \therefore Q. Q. a$ therefore C also shall be a square number. But because $B + C. (D) C \therefore 5.4 \therefore$ not
 b cor. 24. Q. Q. b therefore shall not D be a square number.
 8. Which was to be done.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

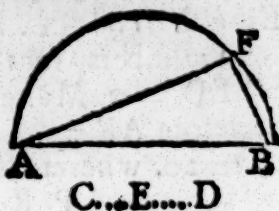
2. Let A be some square number. Take D, E, F, plane numbers unlike, and let D be $\equiv E + F$. make D, E $\therefore A. B.$ and D, F $\therefore A. C.$ I say the thing required is done.

For because D, E + F $\therefore A. B + C$, and $D = E + F$, a therefore shall $A = B + C$. Now suppose
 a 14. 5. B to be square, b then A and B, c and consequent-
 b 21. def. 8. ly D and E are like plane numbers. Which is con-
 c 26. 8. trary to the Hyp.

The same absurdity will follow if C be supposed a square number. Therefore, &c.

PROP. XXX.

To find out two such rational right lines AB, AF, commensurable only in power, as the greater AB shall be in power more than the less AF by the square of a right line BF commensurable in length to the greater.



Let AB be the line given ρ . a Take the square a 1. lem. numbers CD, CE, so that $CD = CE$ (ED) be not 29. 10. Q. b and let there be CD. ED :: ABq. AFq. In a b 3. lem. 10. circle described upon the diameter AB c draw AF, 10. and also BF. Then I say AB, AF, are the lines c 1. 4. required.

For ABq. AFq d :: CD. ED. e therefore ABq $\square$ d constr. AFq. but AB is ρ . f therefore AF is also ρ . But e 6. 10. because CD is Q: and ED not Q: g therefore shall f sch. 12. AB be $\square$ AF. Moreover by reason of the h 10. right angle AFB, is ABq k = AFq + BFq; g 9. 10. therefore seeing ABq. AFq :: CD. ED. by conversion of proportion shall ABq. BFq :: CD. CE :: Q. h 3. 1. 3. Q. l therefore AB $\square$ BF. Which was to be done. k 47. 1. l 9. 10.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9. 5 :: 36. (Q: 6) AFq. then AFq shall be 20. and consequently AF $\sqrt{20}$. therefore BFq = 36 - 20 = 16. wherefore BF is 4.

PROP. XXXI.

To find out two rational lines AB, AF commensurable only in power, so that the greater AB shall be in power more than the less AF by the square of a right line BF incommensurable in length to the greater.



Let AB be the line given ρ . a Take the square a 2. lem 29. numbers CE, ED, so that $CD = CE + ED$ be not 10. Q, and in the rest follow the construction of the preced-prop. I say then the thing required is done.

For

medial proportional line (DE) shall not be a rational number, and so neither shall the excess (DEq) of the square numbers, CEq, CDq. be a square number.

Lemma 2.

2 To find out two such square numbers B, C, as the number compounded of them D is not square. Also to divide a square number A into two numbers B, C, not squares.

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be $\frac{1}{4}$ B, and $D = B + C$. I say the matter is done.

a 24. 8. For B is Q. by the constr. likewise because B.C $:: 1.4 :: Q. Q.$ a therefore C also shall be a square number. But because $B + C. (D) C :: 5.4 ::$ not
b cor. 24. Q. Q. b therefore shall not D be a square number.
8. Which was to be done.

A, 36. B, 14. C, 12. D, 3. E, 2. F, 1.

2. Let A be some square number. Take D, E, F, plane numbers unlike, and let D be $= E + F$. make D, E $:: A. B.$ and D, F $:: A. C.$ I say the the thing required is done.

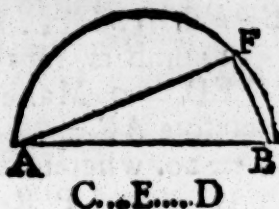
For because D, E + F $:: A. B + C$, and $D = E + F$, a therefore shall $A = B + C$. Now suppose
a 14. 5. B to be square, b then A and B, c and consequent-
b 21. def. 8. ly D and E are like plane numbers. Which is con-
c 26. 8. trary to the Hyp.

The same absurdity will follow if C be supposed a square number. Therefore, &c.

P R O P.

PROP. XXX.

To find out two such rational right lines AB, AF, commensurable only in power, as the greater AB shall be in power more than the less AF by the square of a right line BF commensurable in length to the greater.



Let AB be the line given ρ . a Take the square a 1. lem. numbers CD, CE, so that $CD = CE$ (ED) be not 29. 10. Q. b and let there be $CD : ED :: AB : AF$. In a b 3. lem. 10. circle described upon the diameter AB c draw AF, 10. and also BF. Then I say AB, AF, are the lines c 1. 4. required.

For $AB : AF :: CD : ED$. e therefore $AB : AF :: CD : ED$. d constr. AF but AB is ρ . f therefore AF is also ρ . But e 6. 10. because CD is Q: and ED not Q: g therefore shall f sch. 12. AB be $\perp$ AF. Moreover by reason of the h 10. right angle AFB, is $AB^2 = AF^2 + BF^2$; g 9. 10. therefore seeing $AB : AF :: CD : ED$. by conversion of proportion shall $AB : BF :: CD : CE :: Q$. h 3. 1. 3. Q. I therefore AB $\perp$ BF. Which was to be done. k 47. 1. l 9. 10.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9. 5 :: 36. (Q: 6) AF^2 . then AF^2 shall be 20. and consequently AF $\sqrt{20}$. therefore $BF^2 = 36 - 20 = 16$. wherefore BF is 4.

PROP. XXXI.

To find out two rational lines AB, AF commensurable only in power, so that the greater AB shall be in power more than the less AF by the square of a right line BF incommensurable in length to the greater.



Let AB be the line given ρ . a Take the square a 2. lem 29. numbers CE, ED, so that $CD = CE + ED$ be not 10. Q, and in the rest follow the construction of the preced-prop. I say then the thing required is done.

For

For, as above, AB, AF, are $\dot{p}$ $\square$. also ABq. BFq :: CD. ED. therefore being CD is not Q. AB, BF b shall be $\square$. Which was to be done.

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45. 9 :: 25 (ABq.) 5 (AFq.) therefore AF = $\sqrt{5}$. consequently BFq = 45 - 25 = 20. wherefore BF = $\sqrt{20}$.

P R O P. XXXII.

A _____ To find out two medial lines
B _____ C, D, commensurable only
C _____ in power, comprehending a
D _____ rational rectangle CD, so
that the greater C be more
in power than the lesser D by the square of a right
line commensurable in length to the greater.

a 30. 10. a Take A and B $\dot{p}$ $\square$; so as $\sqrt{Aq - Bq} \square A$.
b 13. 6. b and make A. C :: C. B. c and A. B :: C. D. I say
c 12. 6. the thing is done.

d constr. For because A and d B are $\dot{p}$ $\square$. c therefore shall
e 22. 10. C (f $\sqrt{AB}$) be μ . g and thence also C $\square$ D. b there-
f 17. 6. fore D is likewise μ . Furthermore, whereas A. B
g 10. 10. d :: C. D; and inversely A. C :: B. D :: C. B; and
h 24. 10. Bq is $\dot{p}$. therefore shall CD (k Bq) be $\dot{p}$. Lastly,
k 17. 6. because $\sqrt{Aq - Bq} \square A$, l shall $\sqrt{Cq - Dq}$
l 15. 10. be $\square$ C. therefore, &c. But if $\sqrt{Aq - Bq} \square$
Aq, then shall $\sqrt{Cq - Dq}$ be $\square$ C.

In numbers; let there be A 8, B $\sqrt{48}$ ($\sqrt{64} = 16$) therefore C = $\sqrt{AB} = \sqrt{3072}$. and D = $\sqrt{1728}$. wherefore CD = $\sqrt{5308416} = \sqrt{2304}$.

P R O P. XXXIII.

A _____ To find out two medial lines
D _____ D, E, commensurable in power
B _____ only, comprehending a medial
C _____ rectangle DE, so that the greater
E _____ D shall be more in power than
the less E, by the square of a
right line commensurable to the greater in length.

a Take

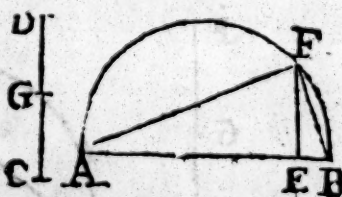
a Take A and C $\bar{p}$ $\bar{q}$, so that $\sqrt{Aq - Cq} \sqsupset A$. a 30. 10.
 b take also B $\bar{q}$ A and C, and make A.D c :: D. B b lem. 21.
 d :: C.E. then D, and E are the lines sought for. 10.

For because A and C e are $\bar{p}$, e and B $\bar{q}$ A and c 13. 6.
 C, f therefore shall B be $\bar{p}$, and D ($\sqrt{AB}$) g shall d 12. 6.
 be μ . But because A.D :: C.E, therefore inversely e constr.
 A.C :: D. E. wherefore seeing A $\bar{q}$ C, therefore D f sch. 12.
 shall be $\bar{q}$ E. therefore E is μ . Furthermore, l be- 10.
 ing D. B :: C. E. and BC is $\mu\nu$. also DE, equal to g 22. 10.
 it, is $\mu\nu$. Lastly, because A. C :: D.E. e seeing $\sqrt{h}$ 10. 10.
 Aq - Cq $\sqsupset A$. therefore $\sqrt{Dq - Eq} \sqsupset D$. there- k 24. 10,
 fore, &c. But if $\sqrt{Aq - Cq} \sqsupset A$. then $\sqrt{Dq - l 22. 10.$
 Eq $\sqsupset$ Eq. m 16. 6.

In numbers, let there be A 8, C $\sqrt{48}$. B $\sqrt{28}$. n 15. 5.
 then D $\sqrt{3072}$. and E $\sqrt{588}$. wherefore D.
 E :: 2. $\sqrt{3}$. and DE = $\sqrt{1344}$.

P R O P. XXXIV.

To find out two right lines
 AF, BF, incommensurable
 in power, whose squares ad-
 ded together make a rational
 figure, and the rectangle
 contained under them medial.



a Let there be found AB, CD, $\bar{p}$ $\bar{q}$; so that $\sqrt{ABq - CDq} \sqsupset AB$ divide CDequally in G. c make
 the rectangle AEB = GCq. Upon AB the diame-
 ter draw a semicircle AFB, erect the perpendicu-
 lar EF, and draw AF, BF. These are the lines
 required. e 17. 6.

For AE.BE d :: BA x AE. AB x BE. But PA x AE
 = AFq; and AB x BE = FBq. f therefore AE. EB
 :: AFq.FBq. therefore being AE g $\sqsupset$ EB, h AFq
 shall be $\sqsupset$ FBq. Moreover ABq (k AFq + FBq)
 l is $\bar{p}\nu$. Lastly EFq l = AEB l = CGq. m therefore
 EF = CG. therefore CD x AB = 2 EF x AB. But CD
 x AB n is $\mu\nu$. o therefore AB x EF, p or AF, FB,
 $\mu\nu$. Which was to be demonstrated.

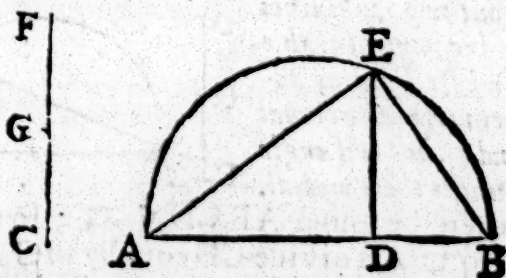
The P sch. 22. 6.

The Explication of the same by-numbers.

Let AB be 6. CD $\sqrt{12}$ then CG $= \sqrt{14} = \sqrt{3}$. But AE $= 3 + \sqrt{6}$. and EB $= 3 - \sqrt{6}$. whence AF shall be $\therefore 18 + 216$. and FB $\sqrt{18} = \sqrt{216}$. Also AFq + FBq is 36, and AF $\times$ FB $= \sqrt{108}$.

But AE is found in this manner. Because BA (6.) AF $\therefore$ AF. AE. therefore 6 AE $=$ AFq $=$ AEq + 3 (EFq.) therefore 6 AE $-$ AEq $=$ 3. Put 3 $=$ e $=$ AE. then $18 + 6e - 9 - 6e - ee$, that is, $9 - ee = 3$. or $ee = 6$. wherefore $e = \sqrt{6}$. and so AE $= 3 + \sqrt{6}$.

P R O P. XXXV.



To find out two right lines AE, EB, incommensurable in power, whose squares added together make a medial figure, and the rectangle contained under them rational.

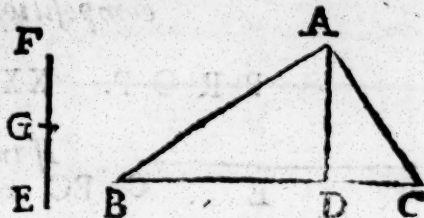
232. 10. a Take AB and CF μ $\square$, so that AB $\times$ CF be ρv , and $\sqrt{ABq - CFq} = AB$. and let the rest be done as in the prec. prop. AE, EB are the lines required.

For, as it is shewn there, AEq $\square$ EBq. also b constr. ABq (AEq + EBq) μv . and lastly AB $\times$ CF b is ρv . c sch. 12. e therefore also AB $\times$ DB, that is, AE $\times$ EB, is 10. ρv . therefore, &c. d sch. 22. 6.

P R O P.

PROP. XXXVI.

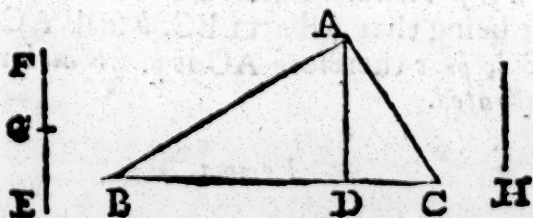
To find out two right lines BA, AC, incommensurable in power, whose squares added together make a medial figure, and the rectangle also contained under them medial, and incommensurable to the figure composed of the squares.



a Take BC and EF μ , so that $BC \times EF$ be $\mu\nu$. a 33. 10. and $\sqrt{BCq - EFq} \perp BC$. and so forward, as in the prec. BA, AC, shall be the lines sought for.

For (as above) $BAq \perp ACq$. also $BAq + ACq$ is $\mu\nu$. and $BA \times BC$ is $\mu\nu$. Lastly, $BC \perp EF$, and c so b const. $BC \perp EG$; likewise $BC \cdot EG d :: BCq \cdot BC \times EG$ c 13. 10. $(BC \times AD, \text{ or } BA \times AC) e$ therefore $BCq (ABq + ACq) d$ 1. 6. $\perp BA \times AC$. therefore, &c. c 14. 10.

Schol.



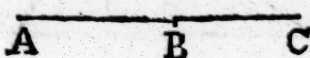
To find out two medial lines incommensurable both in length and power.

a Take BC μ , and let $BA \times AC$ be $\mu\nu$, and $\perp$ a 36. 10. $BCq (BAq + ACq) b$ make BA. H :: H. AC. then b 33. 6. Ifay BC and H are μ $\perp$. For BC is μ . a and $BA \times AC$ (c Hq) is $\mu\nu$. wherefore H is also μ . d Like- c 17. 6. wise $BA \times AC \perp BCq$; therefore Hq $\perp BCq$. d 14. 10. therefore, &c.

Here

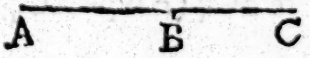
Here begins the senaries of lines irrational by composition.

P R O P. XXXVII.

 If two rational lines AB, BC, commensurable only in power, be added together, the whole line AC is irrational, and is called a binomial line, or of two names.

a hyp. For because AB a $\perp$ BC, thence b shall ACq be
b lem. 26. $\perp$ ABq. But AB a is p. c therefore AC is p. Which
10. was to be demonstrated.
c11.def.10.

P R O P. XXXVIII.

 If two medial lines AB, BC, only in power commensurable, be compounded, and contain a rational rectangle, the whole line AC is irrational, and called a first binomial line.

a hyp. For being that AB a $\perp$ BC, b shall ACq be $\perp$
b lem. 16. ABxBC, p. c therefore AC is p. Which was to be
10. demonstrated.
c11.def.10.

Lemma.

 A rectangle AC, contained under a rational line AB and an irrational line BC, is irrational.
For if the rectangle AC be affirmed p, a then being AB is p, b the breadth BC shall be also p. against the Hyp.

P R O P.

PROP. XXXIX.

If two medial lines AB, BC, commensurable only in power, containing a medial rectangle, be compounded, the whole line AC shall be irrational, and is called a second bimedial line.

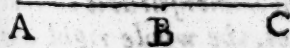


Upon the propound-

ed line DE ρ a make the rectangle DF = ACq; a cor. 16.6. b and DG = ABq + BCq. 47.1. &

Because ABq $\square$ BCq, d therefore ABq + BCq, 11. 6. i. e. DG, $\square$ ABq; but ABq $\square$ is $\mu\nu$, e therefore DG $\square$ hyp. is $\mu\nu$. But the rectangle ABC is taken $\mu\nu$; e and d 16. 10. consequently 2 ABC (f HF) is $\mu\nu$. g therefore EG e 24. 10. and GF are ρ . Being also that DG $\square$ HF; and f 4. 2. DG. HF :: k EG. GF; l therefore EG $\square$ GF. g 23. 10. m therefore the whole EF is ρ . n wherefore the h lem. 26. rectangle DF is $\rho\nu$. o therefore $\sqrt{DF}$, i. e. AC, 10. is ρ . Which was to be demonstrated. k 1. 6. l 10. 10. m 37. 10. n lem. 38. o 11. def.

PROP. XL.



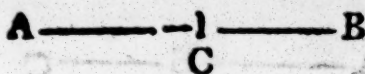
If two right lines AB, BC, commensurable only in power, be added together, making

that which is composed of their squares rational, and the rectangle contained under them medial, the whole right line AC is irrational and is called a Major line.

For whereas ABq + BCq a is $\rho\nu$, and b $\square$ 2 ABC a hyp. $\square$ $\mu\nu$; and so ACq (d ABq + BCq + 2 ABC) e $\square$ b sch. 12. ABq + BCq $\rho\nu$. f therefore shall AC be ρ . Which 10. was to be demonstrated. c hyp. and 24. 10. d 4. 2. e 17. 10.

PROP. III. def. 10.

P R O P. XLI.



If two right lines AC, CB, incommensurable in power, be added together, having that which is made of their squares added together medial, and the rectangle contained under them rational, the whole right line AB shall be irrational, and is called A line containing in power a rational and

a hyp. and a medial rectangle.

f. h. 12. 20. For 2 rectangles ACB a $\rho\nu$, b $\square ACq + CBq$ e $\mu\nu$.
b sch. 12. d therefore 2 ACB d $\square ABq$. wherefore e AB is ρ .
10. Which was to be demonstrated.

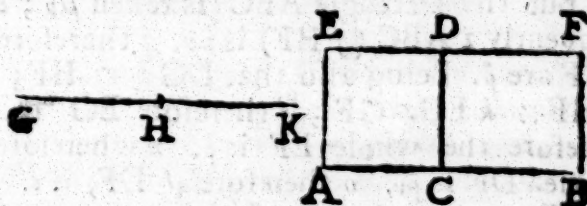
c hyp.

d 17. 10.

e 11. def.

10.

P R O P. XLII.



If two right lines GH, HK, incommensurable in power, be added together, having both that which is composed of their squares medial, and the rectangle contained under them medial, and incommensurable to that which is composed of their squares, the whole right line GK is irrational, and is called A line containing in power two medial figures.

Upon the propounded line FB ρ make the rectangles AF = GKq, and CF = GHq + HKq. Being GHq + HKq (CF) a is $\mu\nu$, the breadth CB b shall be ρ . Also because 2 rectangles GHK (c AD) a is $\mu\nu$, therefore AC b shall be ρ . Moreover because the rectangle AD a $\square CF$, d and AD. CF :: AC. CB, e thence shall e AC be $\square CB$. f wherefore A is g ρ . therefore the rectangle AF. i. e. GKq is $\rho\nu$; b and consequently GK is ρ . Which was to be dem.

a hyp.

b 23. 10.

c 4. 2.

d 2. 6.

e 20. 20.

f 37. 10.

g lem. 38.

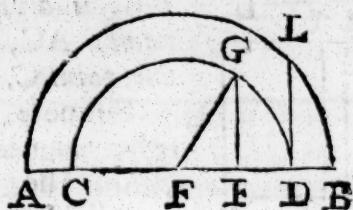
10.

h 21. def.

10.

P R O P.

P R O P. XLIII.

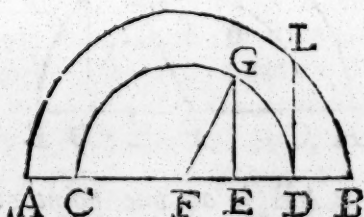


A line of two names, or binominal, AB, can at one point only D be divided into its names, AD, DB.

If it be possible, let the binominal line AB be divided at the point E, into other names AE, EB. It is manifest that the line AB is in both cases divided unequally, since $AD \neq DB$, and $AE \neq EB$.

Because the rectangles ADB, AEB *a* are $\mu\alpha$; *a* and *a* 37. 10.
each of ADq, DBq, AEq, EBq is $\rho\alpha$. *b* and so *b* sch. 27.
ADq + DBq *b* and AEq + EBq are also $\rho\alpha$. *b* there- 10.
fore ADq + DBq = AEq + EBq *c i. e.* 2 AEB = 2 c sch. 5. 2.
ADB is $\rho\nu$. *d* therefore AEB - ADB is $\rho\nu$. there- d sch. 12.
fore $\mu\nu$ exceeds $\mu\nu$ by $\rho\nu$. *e* Which is absurd. 10.
e 27. 10.

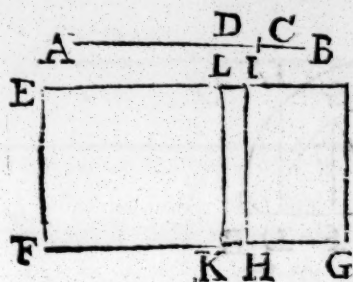
P R O P. XLIV.



A first bimedial line AB is in one point only D divided into its names AD, DB.

Conceive AB to be divided into other names AE, EB, whereupon every one ADq, DBq, EBq, *a* 38. 10.
will be a $\mu\alpha$. and the rectangles ADB, AEB, and *b* sch. 27.
the doubles of them, $\rho\alpha$. *b* therefore 2 AEB = 2 10.
ADB. *c i. e.* ADq + DBq = AEq is $\rho\nu$. Which is abs. c sch. 5. 2.
P P R O P. d 27. 10.

P R O P. XLV.



A second bimedral line AB, is divided into its names AC, CB, only at one point C.

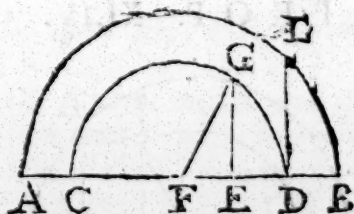
Suppose there were other names AD, DB. Upon the propounded line EF make the rect. angles $EG = ABq$, and

$EH = ACq + CBq$. as also $EK = ADq + DBq$.

- a 39. 10. Because ACq, BCq are $\mu\alpha$ $\square$; b ACq+CBq
 b 16. and (EH) shall be $\mu\nu$. c therefore the breadth FH is ρ .
 24. 10. a moreover the rectangle ACB, d and so 2 ACB
 c 23. 10. (e IG) is $\mu\nu$. c therefore AG is also ρ . And since
 d 24. 10. EH is $f \square$ IG, g and EH IG :: FH. HG. b there-
 e 4. 2. fore FH, HG shall be $\square$. k therefore FG is a bi-
 f lem. 26. nomial, whose names are FH, HG. By the same
 10. reason FG is binomial, and the names of it FK,
 g 1. 6. KG: contrary to the 43. of this Book.

h 10. 10.
 k 37. 10.

P R O P. XLVI.



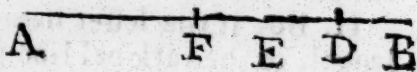
A Major line AB is at one point only D divided into its names AD, DB.

- Imagine other names AE, EB. whereupon the
 a 40. 10. rectangles ADB, AEB, are a $\mu\alpha$. a and as well
 b sch. 27. $ADq + DBq$, as $AEq + EBq$ are $\rho\alpha$. b therefore ADq
 10. $+ DBq = AEq + EBq$, c i. e. $2 AEB - 2 ADB$ is $\rho\nu$.
 c sch. 5. 2. d Which is impossible.
 d 17. 10.

P R O P

P R O P. XLVII.

A line AB contain-
ing in power a ratio-
nal and a medial figure

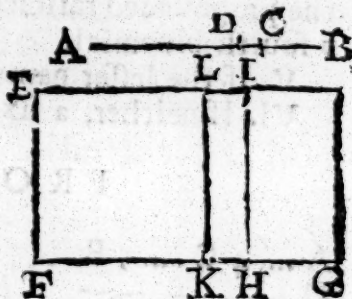


is divided at one point only D into its names AD, DB.

Conceive other names AE, EB. then both AEq
+EBq, and ADq+DBq are $\mu\alpha$. a and the rectangles a 41. 10.
AEB, ADB are $\rho\alpha$. b therefore 2 AEB - 2 ADB, b sch. 27.
c i. e. ADq+DBq $\rightarrow$ AEq + EBq is $\rho\gamma$. d Which 10.
is absurd. c sch. 5. 2.
d 27. 10.

P R O P. XLVIII.

A line AB containing
in power two medial rect-
angles, is at one point on-
ly C divided into its names
AC, CB.



If you would divide
AB into other names
AD, DB. draw upon the
line compounded EF ρ
the rectangles EG=ABq, and EH=ACq + CBq,
and EK=ADq+DBq. then because ACq + CBq,
namely EH, a is $\mu\gamma$, b the breadth FH shall be ρ . a 41. 10.
Also because 2 ACB, c that is, IG, is a $\mu\gamma$, HG b 23. 20.
b shall be likewise ρ . Therefore, whereas EH a $\sqcap$ c 4. 2.
IG. and EH. IG d :: FH.HG, thence FH e shall d 1. 6.
be $\sqcap$ HG. f therefore FG is a binomial, and the e 10. 10.
names of it FH, HG. In like manner FK, KG f 37. 10.
shall be the names of it, against the 43 of this Book.

Second Definitions.

A Rational line being propounded, and the bi-
nomial divided into its names, the greater
of whose names is more in power than the less by
the square of a right line commensurable to the
greater in length; then

I. If the greater name be commensurable in length to the rational line propounded, the whole line is called a binomial line.

II. But if the lesser name be commensurable in length to the rational line propounded, the whole line is called a second binomial.

III. If neither of the names be commensurable in length to the rational line propounded, it is called a third binomial.

Furthermore, if the greater name be more in power than the less, by the square of a right line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded rational line in length, it is called a fourth binomial.

V. If the lesser name be so, a fifth.

VI. If neither, a sixth.

P R O P. XLIX.

a sch. 29. *10.* *b 2.lem.10.* *10.* *c 3.lem.10.* *10.* *d constr.* *e 6.def.10.* *f 6.10.* *g sch. 12.* *10.* *h 9.10.* *k 9.10.* *l 1.def.48.* *10.*

A 4 C 5 B
D —————
E ————— F ————— G
H —————

To find out a first binomial line, EG.
a Take AB, AC, square numbers, whole exceeds CB is not Q. let D be propounded
p. b Take EF $\perp$ D, and *c* make AB.CB :: EFq, FGq. then EG shall be a 1 bin.
For EF *d* $\perp$ D. *e* therefore EF is *p.* *f* also EFq $\perp$ FGq. *g* therefore FG is also *p.* likewise *d* because EFq. FGq :: AB. CB :: not Q. *Q. b* therefore EF $\perp$ FG. Lastly, because by conversion of proportion, EFq.EFq — FGq :: AB. AC :: Q. *Q.* thence EF *k* shall be $\perp \sqrt{EFq - FGq}$. *l* therefore EG is a 1 binominal. Which was to be done.
In numbers thus ; let there be D 8. EF 6. AB 9. CB 5. wherefore because 9. 5 :: 36. 20, therefore FG is $\sqrt{20}$. and consequently EG is $6 + \sqrt{20}$.

P R O P.

PROP. L.

To find out a second binomial line, EG. A 4 C 5 B

Take AB and AC square numbers, the excess of which is CB not Q. Let D be the line propounded $\dot{p}$. D ————
E ———— F ———— G
H ————

Prove it as the prec.

line propounded $\dot{p}$. take FG $\perp$ D, and make CB. AB :: FGq. EFq. then EG will be the line desired.

For FG $\perp$ D. wherefore FG is $\dot{p}$. Also EFq $\perp$ FGq. therefore EF is $\dot{p}$. Likewise because FGq. EFq :: CB. AB :: not Q. Q. thence FG is $\perp$ EF. Lastly, seeing CB. AB :: FGq. EFq, and inversely AB. CB :: EFq. FGq. therefore as in the foregoing Prop. EF $\perp$ $\sqrt{EFq - FGq}$. a whereby EG is a 2 a 2. def. 48. binomial. Which was to be done. 10.

In numbers; let there be D 8, FG 10, AB 9, CB 5. then EF is $\sqrt{180}$. wherefore EG is $10 + \sqrt{180}$.

PROP. LI.

To find out a third binomial line, DF. A 4 C 5 B
L 6

a Take AB, AC, square numbers, the excess of which CB is not Q. and let L be a number not Q. G ————
D ———— F ———— F
H ————

a sch. 19. 10.

next greater than CB, viz. by a unit or two. Let G be the line propounded $\dot{p}$. b Make L. AB :: Gq. b 3. lem. 10. DEq. b and AB. CB :: DEq. EFq. then DF shall be 10. a third binomial.

For because DEq $\perp$ Gq, d DE is $\dot{p}$. also Gq. c constr. 6. DEq :: L. AB :: not Q. Q. e therefore G $\perp$ DE. 10. Likewise being that DEq $\perp$ EFq, d also EF is $\dot{p}$. d sch. 12. Moreover because DEq. EFq :: AB. CB :: not Q. 10. Q. f is DE $\perp$ EF. and being that by constr. and e 6. 10. of equality Gq. EFq :: L. CB :: not Q. Q. (for g L f 9. 10.

P 3

and g sch. 27. 3.

h 9. 10. and CB are not like plane numbers.) *b* therefore shall G be also $\perp$ EF. Lastly, as in the prec. k 3. def. 48. prop. $\sqrt{DEq - EFq} \perp DE$. *k* therefore DF is a 3 binomial. Which was to be done.

In numbers; let there be AB, 9. CB, 5. L, 6. G, 8. then shall be DE $\sqrt{96}$, and EF $\sqrt{4\frac{1}{2}}$. wherefore DF $= \sqrt{96} + \sqrt{4\frac{1}{2}}$.

P R O P. LII.

A ... 3 C 6 B To find out a fourth binomial.
 G _____ line DF.
 a sch. 29. D _____ E _____ F a Take any square number
 10. H _____ AB, and divide it into AC,
 CB not squares. Let G be
 b 2. lem 10 the line propounded $\dot{p}$. *b* take DE $\perp$ G. *c* and
 10. make AB. CB :: DEq. EFq. then DF shall be a 4
 c 3. lem. 10. binomial.
 10.

For, as in the 49 of this Book, DF may be shewn to be a binomial, and also because by constr. and conversion of proportion DEq. DEq - EFq :: AB. AC :: not Q. Q. *d* shall DE be $\perp \sqrt{DEq - EFq}$. *e* therefore DF is a 4 binomial
 d 9. 10. In numbers, let G be 8, DE, 6. then EF shall
 e 4. def. be $\sqrt{24}$. therefore DF is $6 + \sqrt{24}$.
 48. 10.

P R O P. LIII.

A ... 3 C 6 F To find out a fifth binomial
 G _____ line DF.
 D _____ E _____ F Take any square number
 H _____ AB, whose segments AC,
 CB are not Q. Let G be
 the line propounded $\dot{p}$. take EF $\perp$ G. and make
 CB. AB :: EFq. DEq. then shall DF be a 5 bino-
 mial.

For DF shall be a binomial as in the 50. of this Book, and because by construction, and inversion, $DEq. EFq. :: AB. CB$ and so by conversion of proportion, $DEq. DEq - EFq. :: AB. AC :: \text{not } Q.$ *a* therefore shall DE be $\square \sqrt{DEq - EFq.}$ *b* therefore DF is a 5 binomial. Which was to be done.

a 9. 10.
b 5. def. 48.
10.

In numbers, let there be G, 7. EF, 6. then DE shall be $\sqrt{54}$. wherefore DF is $6 + \sqrt{54}$.

P R O P. LIV.

To find out a sixth binomial A 5 C 7 B
line. L 9

Take AC, CB, prime numbers, so that $AD + CB$ (AB) be not Q. take also any number square L. Let G be

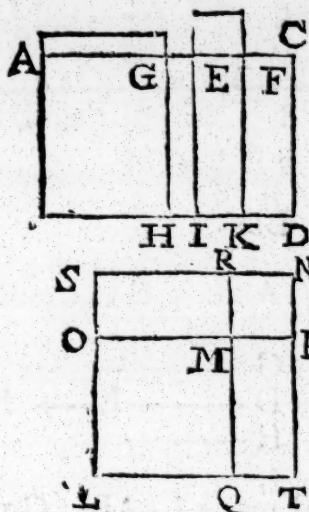
the line propounded *s. a* and make L. $AB :: Gq.$ *a* 3. lem. 10. $DEq.$ and $AB. CB :: DEq. EFq.$ then DF shall be *10.* a 6 binomial.

For DF may be demonstrated binomial as in the 51. of this Book. and also by reason that DE and EF $\square G$. lastly likewise because by constr. and conversion of proportion $DEq. DEq - EFq. :: AB. AC :: \text{not } Q. Q.$ (For AB is prime to AC, *b* and so unlike to it) *c* therefore DE $\square \sqrt{DEq - EFq.}$ *d* therefore DF is a 6 binomial. Which was required.

b sch. 27. 8.
c 9. 10.
d 6. def. 48.
10.

In numbers, let there be G 6. DE $\sqrt{48}$. then EF shall be $\sqrt{28}$. wherefore DF is $\sqrt{48} + \sqrt{28}$.

Lemma.



Let AD be a rectangle, and the side thereof AC divided unequally in E; also let the lesser portion EC be equally divided in F. upon the line AE a make the rectangle AGE = EFq. and from the points G, E, F draw GH, EI, FK, parallel to AB. Let the square LM be made equal to the rectangle AH, and upon OMP produced the square MN = GI. and let the right lines LOS, LQT, NRS, NPT be produced.

I say 1. MS, MT, are rectangles. For by reason of the right angles of the squares OMQ, RMP, a shall QMR be a right line. b therefore RMO, QMP, are right angles. wherefore the parallelograms MS, MT, are rectangles.

2. Hence it is plain that LS c = LT, and consequently that LN is a square.

3. The rectangles SM, MT, EK, FD are equal. For because the rectangle AGE d = EFq. e thence shall AE. EF :: EF. GE. f and so AH. EK :: EK. GI. that is by constr. LM. EK :: EK. MN. g but LM.SM :: SM.MN. therefore EK h = SM k = FD l = MT.

4. Hence LN m = AD.

5. Being that EC is equally divided in F, it is plain that EF, FC, EC are $\square$.

6. If AE $\square$ EC, and AE $\square$ $\sqrt{AEq - ECq}$, o then shall AG, GE, AE, be $\square$. also, because AG.GE :: AH. GI. p therefore shall AH, GI, i.e. LM, MN, be $\square$. Likewise thereupon,

7. OM $\perp$ MP. For by the Hyp. AF $\perp$ EC.
 $\therefore$ therefore EC $\perp$ GE. $\therefore$ wherefore EF $\perp$ GE. q 14. 10.
 but EF. GE :: EK. GI. $\therefore$ therefore EK $\perp$ GI, that 11. 10.
 is, SM $\perp$ MN. but SM.MN :: OM. MP. $\therefore$ there-
 fore OM $\perp$ MP.

8. If AE be supposed $\perp \sqrt{AEq - ECq}$, it is f 19. and
 apparent that AG, GE, AE, are $\perp$. whence LM 17. 10.
 $\perp$ MN. for AG. GE :: AH. GI :: LM. MN.

*These being well considered, we shall easily dispatch
 the six following Propositions.*

P R O P. LV.

*If a space AD be contained under a rational line
 AB, and a first binomial line AC (AE+EC) the right
 line OP which containeth that space in power is irra-
 tional, and called a binomial line.*

All that being supposed which is described and
 demonstrated in the next foregoing Lemma, it is
 manifest that the right line OP containeth in pow-
 er the space AD. a Likewise AG, GE, AE, are a hyp. and
 $\perp$. therefore seeing AE b is $\sqrt{AB}$. c shall also lem. 54 10.
 AG and GE be $\sqrt{AB}$. d therefore the rectangles b hyp.
 AH, GI, that is, the squares LM. MN are $\sqrt{AB}$. c sch. 12. 10
 therefore OM, MP are $\sqrt{AB}$. f and consequently d 20. 10.
 OP is a binomial. Which was to be dem. e lem. 54.

In numbers, let there be AB 5. AC $4 + \sqrt{12}$. 10.
 wherefore the rectangle AD = $20 + \sqrt{300}$ = to f 37. 10.
 the square LN. therefore OP is $\sqrt{15 + \sqrt{5}}$.
 namely a 6 binomial.

P R O P. LVI.

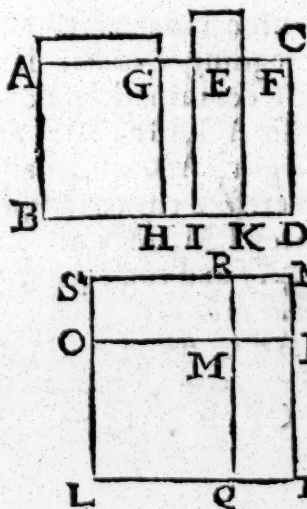
*If a space AD be comprehended under a rational
 line AB, and a second binomial AC (AE + EC) the
 right line OP, which containeth that space AD in
 power, is irrational, and called a first medial line.*

The

The foresaid *Lemma* of the 54. of this Book being
a hyp. and again supposed, then shall OP be $= \sqrt{AD}$. *a* also
lem. 54. 10. AE, AG, GE are $\square$. therefore since AE *b* is ρ
b hyp. $\square AB$, likewise AG, GE *c* shall be $\rho \square AB$.
c sch. 12. 10 therefore the rectangles AH, GI , *i. e.* OMq, MPq .
e lem. 12. *d* are μa . *e* Moreover $OM \perp MP$. Lastly, EF
10. $\square EC$, and EC *f* $\square AB$. *g* wherefore EK , *i. e.*
f hyp. 12. SM , or OMP , is ρv . *b* Consequently OP is a first
10. bimedral. Which was to be demonstrated.
g 20. 10. In numbers, let there be AB 5, and $AC, \sqrt{48}$:
h 38. 10. $+ 6$. then the rectangle $AD = \sqrt{1200 + 30} =$
 OPq . therefore OP is $\sqrt{675 + 75}$. *viz.* a first
bimedral.

See Scheme 57.

PROP. LVII.



If a space AD be contain-
ed under a rational line AB
and a third binomial line AC
 $(AE + EC)$ the right line OP
which containeth in power
the space AD , is irrational,
and called a second bimedral
line.

As above, $OPq = AD$.
also the rectangles AH, GI ,
that is, OMP, MPq are μa .
a Likewise EK or OMP
is μv . *b* therefore OP is a
second bimedral.

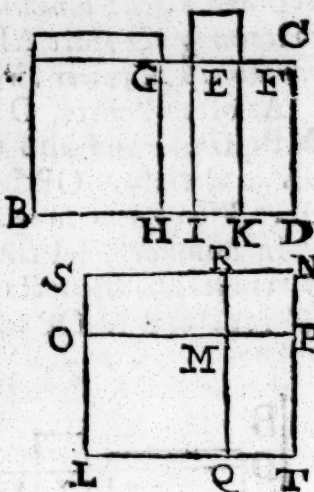
In numbers, let there be
 AB 5. $AC \sqrt{32} + \sqrt{34}$. wherefore AD is $\sqrt{800}$
 $+ \sqrt{600} = OPq$. and so OP is $\sqrt{450 + 50}$.
that is, a second bimedral.

PROP.

P R O P. LVIII.

If a space AD be comprehended under a rational line AB and a fourth binomial AC (AE + EC) the right line OP containing the space AD in power, is that irrational line which is called a Major line.

For again, OMq a $\square$ MPq; and the rectangle AI, i. e. OMq + MPq b is $\rho\nu$. c also EK or OMP is $\mu\nu$. d therefore OP ($\sqrt{AD}$) is a Major line. Which was to be demonstrated.



a lem. 54.
10.
b hyp. and
20. 10.
c hyp. and
22. 10.
d 40. 10.

In numbers, let there be AB 5. and AC $4 + \sqrt{8}$. then the rectangle AD is $20 + \sqrt{200}$. wherefore OP is $\sqrt{20 + \sqrt{200}}$.

P R O P. LIX.

If a space AD be contained under a rational line AB, and a fifth binomial AC, the right line OP which containeth the space AD in power, is that irrational line, which is a line containing a rational and a medial rectangle in power.

Again OMP $\square$ MPq. and the rectangle AI or OMq + MPq is $\mu\nu$. a Likewise the rectangle EK or OMP is $\rho\nu$. b therefore OP ($\sqrt{AD}$) contains in power $\rho\nu$ and $\mu\nu$. Which was to be demonstrated. b 40. 10.

In numbers, let there be AB 5. and AC $2 + \sqrt{8}$. then the rectangle AD = $10 + \sqrt{200}$ = OPq. Wherefore OP is $\sqrt{10 + \sqrt{200}}$.

P R O P.

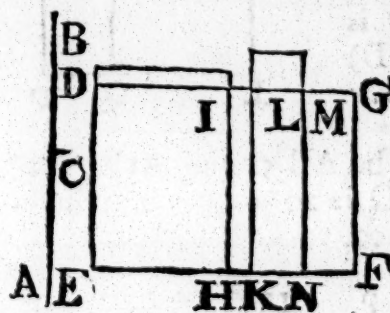
P R O P. LX.

If a space AD be contained under a rational line AB and a sixth binomial AC (AE = EC) the line OP containing the space AD in power is irrational, which containeth in power two medial rectangles.

As often before, OMq $\perp$ MPq, and OMq + MPq is $\mu\nu$. and also the rectangle (EK) OMP is $\mu\nu$. therefore OP = $\sqrt{AD}$ contains in power 2 $\mu\alpha$. Which was to be demonstrated.

In numbers, let there be AB 5. AC $\sqrt{12} + \sqrt{8}$. therefore the rectangle AD or OPq is $\sqrt{300} + \sqrt{200}$. and so OP is $\sqrt{\sqrt{300} + \sqrt{200}}$.

Lemma.



Let a right line AB be unequally divided in C, and let AC be the greater portion, and upon some line DE apply the rectangles DF = ABq, and DH = ACq, and IK = CBq, and let LG, be divided equally in M, and also MN drawn parallel to GF.

I say, 1. The rectangle ACB is = LN or MF.

a 4. 2. and

a For 2 ACB = LF.

3. ax. 1.

2. DL $\perp$ LG. for DK (ACq + CBq) $\perp$ LF (2 ACB) therefore being DK, LF are of equal altitude, c DL shall be $\perp$ LG.

b 7. 2.

c 1. 6.

d 16. 10.

3. If AC $\perp$ CB, d then shall the rectangle DK be $\perp$ ACq and CBq.

e lem. 26.

10.

f 10. 10.

4. Also DL $\perp$ LG. For ACq + CBq e $\perp$ 2 ACB, i. e. DK $\perp$ LF. but DK. LF e :: DL. LG. f therefore DL $\perp$ LG.

g 1. 6.

h 17. 6.

k hyp.

l 10. 10.

m 18. 10.

5. Moreover DL $\perp$ $\sqrt{DLq - LGq}$. For ACq. ACB g :: ACB. CBq. that is, DH. LN :: LN. IK. c wherefore DI. LM :: LM. IL. b therefore DI x IL = LMq. therefore seeing ACq k $\perp$. CBq, that is, DH $\perp$ IK. and l so DI $\perp$ IL. m shall DL be $\perp$ $\sqrt{DLq - LGq}$. Which was to be demonstrated.

6. But

6. But if ACq be but $\square$ CBq, n then shall DL n 19. 10.
be $\square$ $\sqrt{DLq - LGq}$.

The Lemma is preparatory to the 6 following Propo-
sitions.

P R O P. LXI.

The square of a binomial line (AC + CB) applied
unto a rational line DE, makes the latitude DG a
first binomial line.

Those things being supposed, which are descri-
bed and demonstrated in the next preceding Lemma;
because AC, CB a are $\rho \square$, b the rectangle DK a hyp.
shall be $\square$ ACq. c and so DK is ρv . d therefore b lem. 60.
DL $\square$ DE ρ . but the rectangle ACB, and so 2 10.
ACB (LF) e is μv . f therefore the latitude LG is c sch. 12.
 $\rho \square$ DE. g therefore also DL $\square$ LG also DL d 21. 10.
 $\square \sqrt{DLq - LGq}$. from whence k it follows e 22. &
that DG is a first binomial. Which was to be dem. 24. 10.
f 23. 10.
g 13. 10.
h lem. 60.

P R O P. LXII.

The square of a first bimedial line (AC+CB) being
applied to a rational line DE, makes the latitude DG
a second binomial line.

The aforesaid Lemma being again supposed; The
rectangle DK $\square$ ACq. a therefore DK is μv . b a 14. 10.
therefore the latitude DL is $\rho \square$ DE. But be- b 23. 10.
cause the rectangle ACB, and so LF (2 ACB) c is c hyp. and
 ρv . d shall LG be $\rho \square$ DE, e therefore DL, LG sch 22. 10.
are $\square$. f also DL $\square \sqrt{DLq - LGq}$. g from d 2. 10.
whence it is clear that DG is a second binomial. e 13. 10.
Which was to be dem. f lem. 60.
10.
g 2.def.48.
10.

P R O P.

P R O P. LXIII.

The square of a second binomial line ($AC + CB$) applied to a rational line DE makes the breadth DG a third binomial line.

As in the prec. DL is $\rho \sqsubset DE$. Furthermore because the rectangle ACB , and so LF ($2 ACB$) a hyp. and a is $\mu\nu$. b therefore shall LG be $\rho \sqsubset DE$. c 24. 10. Moreover $DL \sqsubset LG$. and also $DL \sqsubset \sqrt{DLq}$ b 23. 10. $- LGq$. d therefore DG is a third binomial. c lem. 60. Which was to be demonstrated.

10.

d 3 def. 48.

20.

P R O P. LXIV.

The square of a Major line ($AB + CB$) applied to a rational line DE , makes the breadth DG a fourth binomial line.

Again $ACq + CBq$. i. e. DK a is $\rho\nu$. b therefore DL is $\rho \sqsubset DE$. also ACB , and so LF ($2 ACB$) b 21. 10. c is $\mu\nu$. d therefore LG is $\rho \sqsubset DE$. e and consequently $DL \sqsubset LG$. Lastly because $AC \sqsubset BC$. c hyp. and f shall DL be $\sqsubset DLq - LGq$. g whence DG is d 23. 10. a fourth binomial. Which was to be demonstrated.

e 13. 10.

f lem. 60.

10.

P R O P. LXV.

The square of a line containing in power a rational and a medial rectangle ($AC + CB$) applied to a rational line DE makes the latitude DG a fifth binomial.

Again, DK is $\mu\nu$. a therefore DL is $\rho \sqsubset DE$. b 21. 10. also LF is $\rho\nu$. b therefore LG is $\rho \sqsubset DE$. c therefore $DL \sqsubset LG$. d likewise $DL \sqsubset \sqrt{DLq} - LGq$. e and so by consequence DG is a fifth binomial. Which was to be demonstrated.

e 5. def.

48. 10.

P R O P. LXVI.

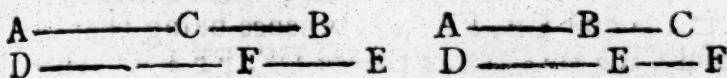
The square of a line containing in power two medial rectangles ($AC + CB$) applied to a rational line DE , makes the latitude DG a sixth binomial line.

A

As before, DL and LG are $\hat{p}$ $\perp$ DE. But for that ACq + CBq (DK) a $\perp$ ACB, b and so DK $\perp$ LF (2 ACB) and also DK. LF $c ::$ DL. LG. d therefore shall DL be $\perp$ LG. e Lastly DL $\perp$ $\sqrt{DLq - LGq}$. f by which it appears that DG is a sixth binomial.

a hyp.
 b 14. 10.
 c 1. 6.
 d 10. 10.
 e lem. 60.
 f 10.
 f 6. def.
48. 10.

Lemma.



Let AB, DE be $\perp$. and make AB. DE $::$ AC. DF.

I say 1. AC $\perp$ DF. as appears by 10. 10. also CB $\perp$ FE. a because AB. DE $::$ CB. FE.

a 19. 5.

2. AC. CB $::$ DF. FE. For AC. DF $::$ AB. DE $::$ CB. FE. therefore inversely AC. CB $::$ DF. FE.

3. The Rectangle ACB $\perp$ DFE. For ACq. ACB

$b ::$ AC. CB $c ::$ DF. EF $::$ DFq. DFE. wherefore by inversion ACq. DFq $::$ ACB. DFE. therefore being ACq $\perp$ DFq. d shall ACB be $\perp$ DFE.

b 1. 6.

c before.

d 10. 10.

4. ACq + CBq $\perp$ DFq + FEq. For because ACq. CBq. $e ::$ DFq. FEq. therefore by addition

e 22. 6.

ACq. + CBq. CBq $::$ DFq + FEq. FEq. therefore being CBq $\perp$ FEq, f shall also ACq + CBq be $\perp$ DFq. + FEq.

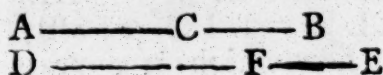
f 10. 10

5. Hence, If ACT $\perp$ or $\perp$ CB, g then likewise shall DE be $\perp$ or $\perp$ EF.

g 10. 10

P R O P.

P R O P. LXVII.



*A line DE, com-
mensurable in length
to a binomial line*

(AC + CB) is it self a binomial line, and of the same order.

Make AB. DE :: AC. DF. *a* then are AC, DF
a lem. 66. $\square$. *a* and CB, FE $\square$. whence being that AC
 10. and CB *b* are $\dot{\rho}$ $\square$, *c* thence DF, FE $\dot{\rho}$ $\square$. there-
b hyp. fore DF is a binomial. But for that AC. CB *a* ::
c lem. 66. DF. FE. if AC $\square$ or $\square \sqrt{ACq - BCq}$, *d* then
 10. and in like manner DF $\square$ or $\square \sqrt{DFq - FEq}$. also
sch. 12. 10. if AC $\square$ or $\square \dot{\rho}$ propounded, *e* then shall
d 15. 10. DF be $\square$ or $\square \dot{\rho}$ propounded. But if CB $\square$
e 12. 10. or $\square \dot{\rho}$. likewise FE $\square$ or $\square \dot{\rho}$. If both AC,
 and 14. 10. CB, $\square \dot{\rho}$. *f* then also both DF, FE, $\square \dot{\rho}$. *g* That
f by def. is, whatloever binomial AB is, DE shall be of
 48. 10. the same order. Which was to be demonstrated.
g 14. 10.

P R O P. LXVIII.

*A line DE commensurable in length to a bimedral
line (AC + CB) is also a bimedral line, and of the
same order.*

Make AB. DE :: AC. DF. *b* therefore AC $\square$
b lem. 66. DF. and CB $\square$ FE. therefore seeing AC and CB
 10. *c* are μ , *d* also DF and FE shall be μ . and for that
c hyp. AC *e* $\square$ CB, *e* therefore FD $\square$ FE. *f* therefore
d 24. 10. DE is 2 μ . Wherefore if the rectangle ACB be $\dot{\rho}\nu$.
e 10. 10. because DFE *b* $\square$ ACB, *g* likewise DFE is $\dot{\rho}\nu$.
f 38. 10. and if that be $\mu\nu$, *b* $\square$ this shall be $\mu\nu$ too. *k* That
g sch. 12. is, whether AB be 1 bimed. or 2 bimed. DF shall
 10. be of the same order. Which was to be demonstrated.
h 24. 10.
k 38. or 39.
 10.

P R O P.

PROP. LXIX.

A ——— C ——— B
D ——— F ——— E

A line DE commensurable to a Major line (AC + CB)

is it self a Major line.

Make AB. DE :: AC. DF. Because AC a $\square$ a hyp. CB, b thence DF $\square$ FE. also ACq + CBq a is ρv . blem. 66. and so being DFq + FEq b $\square$ ACq + CBq, c also 10. DFq + FEq is ρv . lastly, the rectangle ACB a is c sch. 12. μv . d therefore the rectangle DFE is μv . (because 10. DFE is b $\square$ ACB) e wherefore DE is a Major d 24. 10. line. Which was to be demonstrated. e 40. 10.

PROP. LXX.

A line DE commensurable to a line containing in power a rational and a medial rectangle (AC + CB) is a line containing in power a rational and a medial rectangle.

Again make AB. DE :: AC. DF. Because AC a $\square$ CB, b also DF $\square$ FE. likewise because a hyp. ACq + CBq a is μv , c therefore DFq + FEq shall blem. 66. be μv . lastly because the rectangle ACB a is ρv , d 10. also DFE is ρv . Therefore DE contains in power c 24. 10. ρv and μv . Which was to be demonstrated. d sch. 12. 10. e 41. 10.

PROP. LXXI.

A ——— C ——— B
D ——— F ——— E

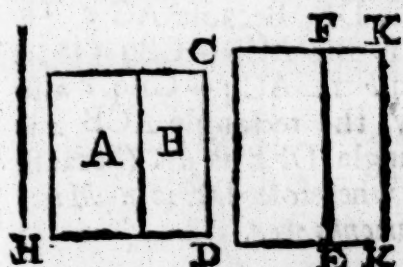
A line DE commensurable to a line containing two medial rectangles in power (AC + CB) is also a line containing in power two medial rectangles.

Divide DE, as in the prec. Because ACq a $\square$ a hyp. CBq, b thence shall DFq be $\square$ FEq. also for that b lem. 66. ACq + CBq a is μv , c shall DFq + FEq be also μv . 10. And in like manner because ACB a is μv , d also c 24. 10. DFE is μv . Lastly, because ACq + CBq $\square$ ACB, d 24. 10. e shall

Q

e 14. 10. e shall $DFq + FEq$ be $\sqsubset DFE$. f From whence
 f 42. 10. it follows that DE contains in power 2 $\mu\alpha$. Which
 was to be demonstrated.

P R O P. LXXII.



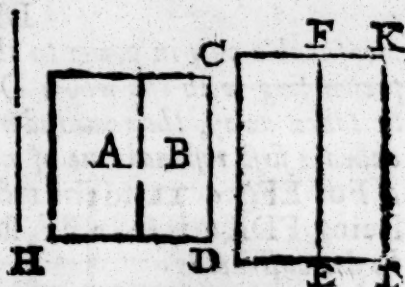
If a rational rectan-
 gle A and a medial B,
 be composed together,
 these four irrational
 lines will be made;
 either a binomial, or a
 first bimedial, or a ma-
 jor, or a line contain-

ing in power a rational and a medial rectangle.

Namely, if $Hq = A + B$, then H shall be one
 of the four lines which the Theoreme mentions.
 For upon CD the propounded ρ^c , a make the
 rectangle $CE = A$, and $FI = B$. b and so $CI =$
 a cor. 16. 6. Hq . Whereas then is A ρ^v , likewise CE is ρ^v . c
 b 2. ax. 1. therefore the latitude CF is $\rho^c \sqsubset CD$. and be-
 c 21. 10. cause B is μv , also FI shall be μv . d therefore FK
 d 23. 10. is $\rho^c \sqsubset CD$. e therefore CF, FK are $\rho^c \sqsubset$. and
 e 13. 10. so the whole CK f is binom. wherefore if $A \sqsubset B$,
 f 37. 10. i. e. $CE \sqsubset FI$, g then $CF \sqsubset FK$. therefore if CF
 g 1. 6. $\sqsubset \sqrt{CFq - FKq}$, h likewise CK shall be a 1.
 h 11. def. 48. bin. and consequently $H = \sqrt{CI}$ k is a bin. If
 i 10. CF be supposed $\sqsubset \sqrt{CFq - FKq}$, l then shall
 k 55. 10. CK be a 4. bin. wherefore $H(\sqrt{CI})$ m is a major
 l 4. def. 48. line. But if $A \sqsupset B$, g then shall CF be $\sqsupset FK$.
 m 10. consequently if $FK \sqsupset \sqrt{FKq - CFq}$, n then shall
 n 58. 10. CK be a 2. bin. o wherefore H is a first 2 μ . lastly
 n 2. def. 48. if $FK \sqsupset \sqrt{FKq - CFq}$, p then CK shall be a
 o 10. fifth binom. q whence H shall contain in power ρ^v
 p 56. 10. and μv . Which was to be demonstrated.
 q 5. def. 48.
 r 10.
 s 19. 10.

P R O P. LXXIII.

If two medial rectangles, A, B, incommensurable to one another be composed together, the two remaining irrational lines are made, either a second bimedial, or a line containing in power two medial rectangles.



As H containing in power $A + B$ is one of the said irrational lines. For upon CD propounded p^c draw the rectangle $CE = A$, and $FI = B$. whence $Hq = CI$. Therefore because CE and FI a are μa . b the latitudes CF, FK, shall be p^c $\square$ CD. also because $CE a \square FI$, and $CE. FI c :: CF. FK$, d therefore $CF \square FK$. e therefore CK is a 3 bin. namely, if $CF \square \sqrt{CFq} - FKq$. whence $H = \sqrt{CI}$ f shall be a second 2μ . But if $CF \square \sqrt{CFq} - FKq$, g then CK shall be a 6 binom. b and consequently H contains in power $2 \mu a$. Which was to be demonstrated.

a hyp.
 b 23. 10.
 c 1. 6.
 d 10. 10.
 e 3. def. 48.
 f 10.
 g 57. 10.
 h 6. def. 48.
 $10.$
 h 60. 10.

Here begins the Senaires of lines irrational by Subtraction.

P R O P. LXXIV.

D — E — F If from a rational line DF a rational line DE, commensurable in power only to the whole DF, be taken away, the residue EF is irrational, and is called an Apotome or residual line.

For $EFq a \square DEq$; b but DEq is $p^c v$; c therefore EF is p^c . Which was to be demonstrated.

In numbers; let there be DF, 2. DE, $\sqrt{3}$. then EF shall be $2 - \sqrt{3}$.

a lem. 26.
 $10.$
 b hyp.
 c 0. 6. 11.
 $def.$ 10.

Q₂

P R O P.

The tenth Book of

P R O P. LXXV.

D ——— E ——— F If from a medial line
DF a medial line DE com-
mensurable only in power to the whole DF, and com-
prehending with the whole DF a rational rectangle,
be taken away, the remainder EF is irrational, and is
called a first residual line of a medial.

a sch. 16. For EFq a $\perp$ to the rectangle FDE. therefore
10. seeing FDE b is $\rho\nu$. c EF shall be ρ . Which was to
b hyp. be demonstrated.

c 20. and In numbers, let DF be $v\sqrt{54}$, and DE $v\sqrt{24}$.
11. def. 10. therefore EF is $v\sqrt{54} - v\sqrt{24}$.

P R O P. LXXVI.

D ——— E ——— F If from a medial line DF, a
medial line DE, be taken away
being commensurable only in power to the whole DF,
and comprehending together with the whole line DF
a medial rectangle, the remainder EF is irrational,
and is called a second residual of a medial line.

a hyp. Because DFq and DEq a are $\mu\alpha$ $\perp$, b therefore
b 16. 10. shall DFq + DEq be $\perp$ DEq. c wherefore DFq
c 24. 10. + DEq is $\mu\nu$. also the rectangle FDE, and so 2
d cor. 7. 1. FDE, a is $\mu\nu$. therefore EFq (d DFq + DEq -
e 27. 10. 2 FDE) e is $\rho\nu$. wherefore EF is ρ . Which was to be
demonstrated.

In numbers, let DF be $v\sqrt{18}$. and DE $v\sqrt{8}$.
then EF $v\sqrt{18} - v\sqrt{8}$.

P R O P. LXXVII.

A ——— B ——— C If from a right line AC be
taken away a right line AB
being incommensurable in power to the whole BC, and
making with the whole AC that which is composed of
their squares rational, and the rectangle contained
under them medial, the remainder BC is irrational, and
is called a Minor line.

a hyp. For ACq + ABq a is $\rho\nu$. but the rectangle ACB
b sch. 12. a is $\mu\nu$. b. therefore 2 CAB $\perp$ ACq + ABq (2 c CAB
10. +
c 7. 2.

+BCq.) *d* therefore ACq+ABq $\perp$ BCq. *e* there- *d* 17. 10.
fore BC is $\bar{p}$. Which was to be demonstrated. *e* 11. def.

In numbers, let AC be $\sqrt{18} + \sqrt{108}$; AB $\sqrt{10}$.
: $18 - \sqrt{108}$. then BC is $\sqrt{18 - \sqrt{108} - \sqrt{10}}$.
 $18 - \sqrt{108}$.

PROP. LXXVIII.

D ——— E ——— F If from a right line DF
be taken away a right line
DE, being incommensurable in power to the whole line
DF, and with the whole DF making that which is com-
posed of their squares medial, and the rectangle con-
tained under the same lines rational, the line remain-
ing EF is irrational, and is called a line making a
whole space medial with a rational space.

For $\angle$ FDE *a* is $\bar{p}v$. *b* and DFq + DEq is μv . *c* a hyp. $\mathcal{E}$
therefore $\angle$ FDE $\perp$ DFq + DEq *d* ($\angle$ FDE + *sch.* 12. 10.
EFq) *e* therefore EF is $\bar{p}$. Which was to be dem. *b* hyp.

In numbers, let DF be $\sqrt{216} + \sqrt{72}$; DE *c* *sch.* 12.
 $\sqrt{216} - \sqrt{72}$. therefore EF is $\sqrt{216 - \sqrt{72} - \sqrt{10}}$.
 $\sqrt{72} - \sqrt{10}$. *d* 7. 2.
esch. 12. 10.
and 11. def.
10.

PROP. LXXIX.

D ——— E ——— F If from a right line DF be
taken away a right line DE,
incommensurable in power to the whole DF, and which
together with the whole makes that which is composed
of their squares medial, and the rectangle contained
under them also medial and incommensurable to that
which is composed of their squares, the remainder is
irrational, and is called a line making a whole space
medial with a medial space.

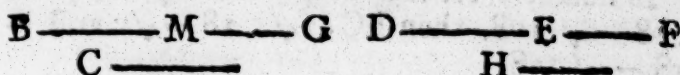
For $\angle$ FDE, and FDq + DEq *a* are μa ; *b* there- a hyp. $\mathcal{E}$
fore EFq ($\angle$ DFq + DEq - $\angle$ FDE) is $\bar{p}v$. *d* and so 24. 10.
consequently EF is $\bar{p}$. Which was to be dem. *b* 27. 10.

In numbers; let DF be $\sqrt{180} + \sqrt{60}$. DE *c* *cor.* 7. 2.
 $\sqrt{180} - \sqrt{60}$. then EF shall be $\sqrt{180 - \sqrt{60} - \sqrt{10}}$. *d* 11. def.
 $\sqrt{60} - \sqrt{10}$. 10.

Q 3

Lem.

Lemma.



If there be the same excess between the first magnitude BG and the second C (MG) as is between the third magnitude DF and the fourth H (EF); then alternately, the same excess shall be between the first magnitude BG and the third DF, as is between the second C and the fourth H.

a hyp.

b 15. ax. 1.

For because that a to the equals BM, DE, are added the equals MG, EF, that is, C, H; the excess of the wholes BG, DF, b shall be equal to the excess of the parts added C, H. Which was to be dem.

Coroll.

Hence, Four magnitudes Arithmetically proportional, are alternately also Arithmetically proportional.

P R O P. LXXX.

A ——— B — D ——— C To an Apotome or residual line AB only one rational right line BC, being commensurable in power only to the whole AB, is congruent, or can be joyned.

If it be possible, let some other line BD be added to it; a then the rectangles ACB, ADB, b and so consequently the doubles of them are ua . c cor. 7.2. wherefore seeing $ACq + BCq - 2 ACB c = ABq$ d lem. 79. $c = ADq + DBq - 2 ADB$. therefore alternately $ACq + BCq - ADq + BDq d = 2 ACB - 2 ADB$. But $ACq + BCq - ADq + BDq e$ is e hyp. and $2 ACB - 2 ADB$ is p. f therefore $2 ACB - 2 ADB$ is p. v. Which is f sch. 12. 10. absurd. g 27. 10.

P R O P.

P R O P. LXXXI.

A—B—D—C

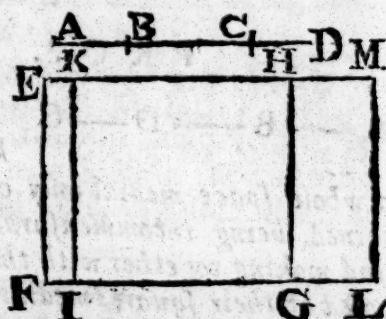
To a first medial residual line AB only one medial right line BC, being commensurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joyned.

Conceive BD to be such a line as may be joyned to it; then because ACq and BCq, as well as ADq and BDq *a* are $\mu\alpha$ $\square$. *b* also ACq + BCq, and ADq + BDq shall be $\mu\alpha$. *c* but the rectangles ACB, ADB, *d* and so 2 ACB and 2 ADB are $\rho\alpha$. *e* therefore 2 ACB = 2 ADB, *f* that is, ACq + BCq = ADq + BDq is $\rho\nu$. *g* Which is absurd.

a hyp.
b 10. and
24. 10.
c hyp.
d sch. 12.
10.
e sch. 17.
10.
f 7. 2. and
lem 79. 10.
g 27. 10.

P R O P. LXXXII.

Unto a second medial residual line AB only one medial right line BC, commensurable only in power to the whole, and with it containing a medial rectangle, can be joyned.



If it be possible, let some other line BD be added to it; and upon EF ρ make the rectangle EG = ACq + BCq; as also the rectangle EL = ADq + BDq. likewise EI = ABq. Now 2 ACB + ABq = ACq + BCq = EG. therefore seeing EI = ABq, *a* also KG shall be = 2 ACB. more. over ACq and BCq *b* are $\mu\alpha$ $\square$. *c* therefore EG (ACq + BCq) is $\mu\nu$. *d* therefore the breadth EH is ρ $\square$ EF. *e* Further, the rectangle ACB *f* and so 2 ACB (KG) is $\mu\nu$. *d* therefore KH is also B, ρ $\square$ EF. lastly, because ACq + BCq (EG) *g* $\square$ 2 ACB (KG) and EG. KG *b* :: EH, KH. *k* therefore EH $\square$ HK. *l* therefore EK is a residual line, *k*

Q 4

whereto 174 10.

whereto KH is congruent. by the same reason also shall KM be congruent to the said EK. Which is repugnant to the 80. prop. of this Book.

P R O P. LXXXIII.

A — B — D — C To a Minor line AB only one right line BC can be joined being incommensurable in power to the whole, and making together with the whole line that which is composed of their squares rational, and the rectangle which is contained under them medial.

Conceive any other BD to be congruent to it; Therefore whereas ACq + BCq, and ADq + BDq are ρ^a . their excesses ($2b$ ACB — : 2 ADB) c is ρ^v . Which is absurd; because ACB and ADB are μa by the Hyp.

i hyp.
b lem. 97.
10.
c sch. 17.
10.
d 27. 10.

P R O P. LXXXIV.

A — B — D — C Unto a line (AB) making with a rational space a whole space medial only one-right line BC can be joined, being incommensurable in power to the whole, and making together with the whole that which is composed of their squares medial, and the rectangle which is contained under them rational.

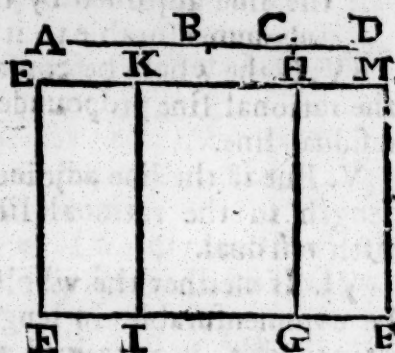
Suppose some other BD to be congruent also to it; a then the rectangles ACB, ADB, b and so 2 ACB and 2 ADB are ρ^a . therefore 2 ACB — : 2 ADB, c that is, ACq + BCq — : ADq + BDq d is ρ^v . Which is absurd; since ACq + BCq, and ADq + BDq are μa by the Hyp.

a hyp.
b sch. 12.
10.
c lem. 79.
10.
d sch. 27.
10.

P R O P.

P R O P. LXXXV.

To a line AB, which with a medial space makes a whole space medial, can be joined only one right line BC, incommensurable in power to the whole, and making with the whole both that which is composed of their squares medial, and the rectangle which is



contained under them medial and incommensurable to that which is composed of their squares.

Those things being supposed which are done and shewn in the 82 prop. of this Book; it is clear that EH and KH are $\perp$ EF. Besides, being that $AC^2 + CB^2$, that is, the rectangle EG, *a* is $\perp$ *a hyp.* ACB. *b* and so EG $\perp$ 2 ACB (KG;) and EG. *b* 14. 12. KG *c* :: EH. KH; shall EH be $\perp$ KH. therefore *c* 1. 6. EK is a residual line, and the line congruent to it is KH. In like manner may KM be shewn to be congruent to the residual EK, against the 80. prop. of this Book.

Third Definitions.

A Rational line and a residual being propounded, if the whole be more in power than the line joined to the residual, by the square of a right line commensurable unto it in length; then

I. If the whole be commensurable in length to the rational line propounded, it is called a first residual line.

II. But if the line adjoined be commensurable in length to the rational line propounded, it is called a second residual line.

III. If neither the whole nor the line adjoined be commensurable in length to the rational line propounded, it is called a third residual line.

Moreover

Moreover, if the whole be more in power than the line adjoined by the square of a right line incommensurable to it in length, then

IV. If the whole be commensurable in length to the rational line propounded, it is called a fourth residual line.

V. But if the line adjoined be commensurable in length to the rational line propounded, it is a fifth residual.

VI. If neither the whole nor the line adjoined be commensurable in length to the rational line propounded, it is termed a sixth residual line.

P R O P. LXXXVI, 87, 88, 89, 90, 91.

A 4 C 5 B

D -----

E ----- F

G

H -----

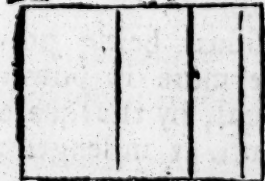
To find out a first, second, third, fourth, fifth, and sixth residual line.

Residual lines are found out by subducting the less names or parts of binomials

from the greater. Ex. gr. Let $6 + \sqrt{20}$ be a first binomial then shall $6 - \sqrt{20}$ be a first residual. So that it is not necessary to repeat more concerning the finding of them out.

Lemma.

A D FGE



B L C K H I

T N P

S V X

Q R M

O

Let AC be a rectangle contained under the right lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F. and let the rectangle AGE be = FEq. and the rectangles AI, DK, FH, finished. Then let the square LM = AH be made, and square NO = GI; and the lines NSR, OST, produced.

I say, 1. The rectangle AI = LM + NO = TOq + SOq. which appears by the constr.

2. The rectangle DK = LO.

For because the rectangle AGE = FEq. b thence are AG, FE, GE, $\frac{1}{2}$ c and so AH,

a constr.

b 17. 6.

c 1. 6.

AH, FI, GI $\therefore$, a that is, LM, FI, NO, $\therefore$; but LM, LO, NO d are $\therefore$; therefore FI = e LO f = d sch. 32.6: DK = g NM. e 9. 5. f 36. 1. g 43. 1.

3. Hence, AC = AI - DK - FI = LM + NO - LO - NM = TR. h 16. 10.

4. It is manifest that DF, FE, DE, are $\perp$. k 18. and

5. If AE $\perp$ DE, and AE $\perp$ $\sqrt{AEq - DEq}$, then shall AG, GE, AE be $\perp$. 10. 10. l hyp.

6. Also, because AE $\perp$ DE, m thence shall AE, FE be $\perp$. n and so AI, FI, that is, LM + NO and LO are $\perp$. m 13. 10. n 1. 6. &

7. Because AG * $\perp$ GE, n shall AH, GI, that is, LM, NO be $\perp$. * before. 10. 10.

8. But because AE $\perp$ DE, o therefore shall FE, GE be $\perp$, n and so the rectangle FI $\perp$ GI, that is, LO $\perp$ NO. wherefore seeing LO. NO p :: TS. p 2. 6. q 10. 10.

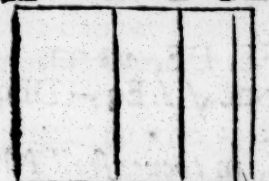
9. If AE be put $\perp$ $\sqrt{AEq - DEq}$, then shall AG, GE, AE be $\perp$. 17. 10.

10. f Wherefore the rectangles AH, GI, that is, TOq, SOq shall be $\perp$. f 1. 6. and 10. 10.

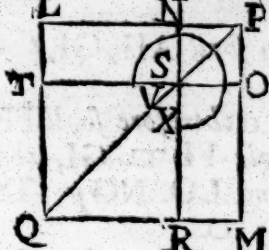
PROP.

P R O P. XCII.

A D F G E



B L C K H I



Q R M

If a space AC be contained under a rational line AB, and a first residual line AD (AE—DE) the right line TS, which containeth the space AC in power, is a residual line.

Use the foregoing Lemma for a preparatory to the demonstration of this prop. Therefore $TS = \sqrt{AC}$. Also AG, GE, AE, are $\perp$; therefore since $AE \perp a AB \hat{p}$, b also AG and GE shall be $\perp AB$, c therefore the rectangles AH, and GI, that is, TO q and SO q are $\hat{p}a$. d Likewise TO, SO, are $\hat{p} \perp$. e and consequently TS is a residual line. Which was to be demonstrated.

a hyp.

b 12. 10.

c 20. 10.

d lem. 91.

10.

e 74. 10.

are $\hat{p}a$. d Likewise TO, SO, are $\hat{p} \perp$. e and consequently TS is a residual line. Which was to be demonstrated.

P R O P. XCIII.

See the prec. Scheme.

If a space AC be contained under a rational line AB and a second residual AD (AE—DE) the right line TS, containing the space AC in power, is a first medial residual line.

a hyp.

b 13. 10.

c 22. 10.

d lem. 74

10.

e hyp.

f 20. 10.

g 75. 10.

Again, by the foregoing Lemma, AG, GE, AE are $\perp$. therefore a since AE is $\hat{p} \perp AB$, b also AG, GE, shall be $\hat{p} \perp AB$. c therefore the rectangles AH, GI, that is, TO q , SO q are μa . d likewise TO $\perp$ SO. Lastly, because DE $e \perp AB \hat{p}$, f the right angle DI, and the half thereof DK or LO, that is, TOS shall be $\hat{p}v$. g from whence it follows that TS ($\sqrt{AC}$) is a first medial residual. Which was to be demonstrated.

P R O P.

PROP. XCIV.

See Scheme 92.

If a space AC be contained under a rational line AB and a third residual AD (AE — DE) the right line TS containing in power the space AC is a second medial residual line.

As in the former, TO and SO are μ . Therefore because DE a is $\rho \sqcap AB$, b the rectangle DI, c and so DK, or TOS, shall be $\mu\nu$. therefore $TS = \sqrt{AC}$ is a second medial residual. Which was to be dem.

a hyp.
 b 21. 10.
 c 24. 10.
 d 76. 10.

PROP. XCV.

See Scheme 92.

If a space AC be contained under a rational line AB and a fourth residual AD (AE — DE) the right line TS containing the space AC in power, is a Minor line.

As before, TO $a \sqcap$ SO. Therefore because AE b is $\rho \sqcap AB$, c shall AI (TOq + SOq) be $\rho\nu$. but, as before, the rectangle TOS is $\mu\nu$. d therefore $TS = \sqrt{AC}$ is a Minor line. Which was to be dem.

a lem. 91.
 b 10.
 c 20. 10.
 d 77. 10.

PROP. XCVI.

See Scheme 92.

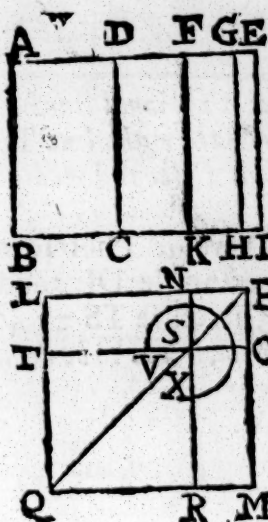
If a space AC be contained under a rational line AB and a fifth residual AD (AE — DE) the right line TS containing in power the space AC, is a line which maketh with a rational space the whole space medial.

For again TO $a \sqcap$ SO. therefore since AE a is $\rho \sqcap AB$ b also AI, that is, TOq + SOq shall be $\mu\nu$. But, as in the 93. the rectangle TOS is $\rho^2\nu$. c whence $TS = \sqrt{AC}$ is a line which with $\rho^2\nu$ makes a whole $\mu\nu$. Which was to be dem.

a hyp.
 b 22. 10.
 c 78. 10.

PROP.

P R O P. XCVII.

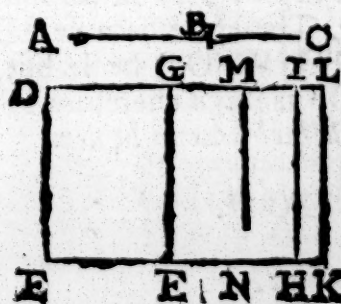


If a space AC be contained under a rational line AB, and a sixth residual AD ($AE - DE$) the right line TS containing in power the space AC is a line making with a medial rectangle, a whole space medial.

As often above, $TO \perp SO$. also, as in 96. $TOq + SOq$ is $\mu\nu$. but the rectangle TOS is $\rho\nu$, as in 94. *a* Lastly, $TOq + SOq \perp TOS$. *b* therefore $TS = \sqrt{AC}$ is a line which with $\mu\nu$ makes a whole $\mu\nu$. Which was to be demonstrated.

a lem. 91.
io.
b 79. 10.

Lemma.



Upon a right line DE * apply the rectangles $DF = ABq$, and $DH = ACq$, and $IK = BCq$. and let GL be bisected in M, and the line MN drawn parallel to GF.

Then 1. The rectangle DK is $= ACq + BCq$. as the construction manifests.

2. The rectangle ACB $= GN$ or MK . For $DK = ACq + BCq$ *b* $= 2 ACB + ABq$. but $ABq = DF$. therefore GK *c* $= 2 ACB$. and consequently GN or $MK = ACB$.

3. The rectangle $DIL = MLq$. For because $ACq : ACB$ *e* $:: ACB : BCq$, that is, $DH : MK :: MK : IK$. *e* thence is $DI : ML :: ML : IL$. *f* therefore $DIL = MLq$.

4. If AC be taken $\perp$ BC, then DK shall be $\perp$ ACq. For $ACq + BCq$ (DK) *g* $\perp ACq$.

5. Like

a constr.

b 7. 2.

c 3. ax. 1.

d 7. ax. 1.

e 1. 6.

f 17. 6.

g 16. 10.

5. Likewise $DL \sqsupset \sqrt{DLq - GLq}$. For because $DH (ACq) \sqsupset IK (BCq)$ *b* thence shall DI *h* 10. 10. be $\sqsupset IL$. *k* therefore $\sqrt{DLq - GLq} \sqsupset DL$. *k* 18. 10.

6. Also $DL \sqsupset GL$. For $ACq + BCq \sqsupset l$ 2 *l* lem. 26. ACB . that is, $DK \sqsupset GK$. *m* therefore $DL \sqsupset$ 10. *GL*. *m* 10. 10.

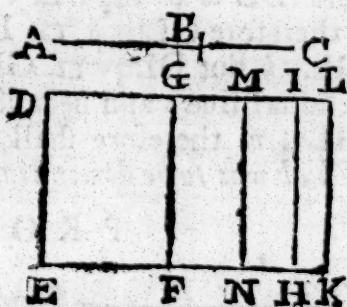
7. But if AC be taken $\sqsupset BC$, then DL shall *n* 19. 10. be $\sqsupset \sqrt{DLq - GLq}$.

P R O P. XCVIII.

The square of a residual line $AB (AC - BC)$ applied to a rational line DE , makes the breadth DG a first residual line.

Do as is enjoined in the Lemma next preceding. Then because AC , BC , *a* are $\dot{p}$ $\sqsupset$. *b* also

$DK (ACq + BCq)$ shall be $\sqsupset ACq$. *c* therefore 10. DK is $\dot{p}v$. *d* wherefore DL is $\dot{p} \sqsupset DE$. *e* Like. *cf* sch. 12. 10. *w* the rectangle GK (2 ACB) is μv . *f* therefore *d* 21. 10. GL is $\dot{p} \sqsupset DE$. *g* and consequently $DL \sqsupset GL$. *e* 22. and *b* But $DLq \sqsupset GLq$. *k* therefore DG is a residual, 24. 10. *l* and that of the first order (because *m* $AC \sqsupset BC$, *f* 23. 10. and therefore $DL \sqsupset \sqrt{DLq - GLq}$.) Which *g* 13. 10. *h* *sch.* 12. 10. *k* 74. 10. *l* 1. *def.* 85. 10. *m* lem. 97. 10.



P R O P.

P R O P. XCIX.

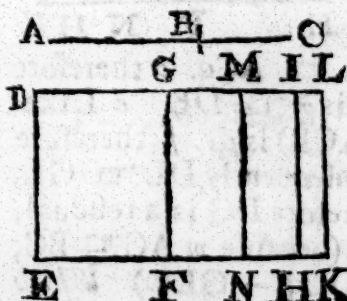
See the following Scheme.

The square of a first medial residual line AB (AC — BC) applied to a rational line DE, makes the breadth DG a second residual line.

Supposing the foregoing Lemma; because AC and BC are $\mu \sqcup$, b thence shall DK (ACq + BCq) be $\sqcup$ ACq. c wherefore DK is $\mu \nu$. d therefore DL is $\dot{\rho} \sqcup$ DE. e also GK (2 ACB) is $\dot{\rho} \nu$. f therefore GL is $\dot{\rho} \sqcup$ DE; g wherefore DL $\sqcup$ GL. h But DLq $\sqcup$ GLq. k therefore DG is a residual line: and because DL is $\sqcup \sqrt{DLq - GLq}$. m therefore shall DG be a second residual. Which was to be demonstrated.

P R O P. C.

a 23. 10.
 b lem. 97.
10.
 c 24. 10.
 d 23. 10.
 e hyp. and
sch. 12. 10.
 f 21. 19.
 g 13. 10.
 h sch. 12.
10.
 k 74. 10.
 l lem. 97.
10.
 m 2. def.
85. 10.



The square of a second medial residual line AB (AC — BC) applied to a rational line DE, makes the breadth DG a third residual line.

Again DK is $\mu \nu$. a wherefore DL is $\dot{\rho} \sqcup$ DE. also DGK is $\mu \nu$. b whence GL is $\dot{\rho} \sqcup$ DE. c likewise DK $\sqcup$ GK. d wherefore DL $\sqcup$ GL. e but DLq $\sqcup$ GLq. f therefore DG is a residual line, and that of the third order, g because DL $\sqcup \sqrt{DLq - GLq}$. Which was to be demonstrated.

P R O P. CI.

a 23. 10.
 b lem. 97.
10.

See the following Scheme.

The square of a Minor line AB (AC — BC) applied to a rational line DE, makes the breadth DG a fourth residual.

As before, $ACq + BCq$, that is DK , is $\dot{p}v$. *a* therefore DL is $\dot{p} \sqcup DE$. but the rectangle ACB , and $\ast$ *hyp.* so GK ($2 ACB$) $\ast$ is μv . *b* wherefore GL is $\dot{p} \sqcup DE$. *c* therefore $DL \sqcup GL$. *d* but $DLq \sqcup GLq$. *c* $13. 10.$ and because $\ast ACq \sqcup BCq$, *e* thence shall DL be d *sch. 12.* $\sqcup \sqrt{DLq - GLq}$. *f* therefore DG has the conditions required to a fourth residual. Which was *e* *lem. 97.* to be demonstrated. *10.*

P R O P. CII.

f $4. def. 85.$
10.

See Scheme 100.

The square of a line AB ($AC - BC$) which makes with a rational space the whole space medial, applied to a rational line DE , makes the breadth DG a fifth residual line.

For, as above, DK is μv . *a* wherefore DL is $\dot{p} \sqcup DE$. also GK is $\dot{p}v$. *b* whence GL is $\dot{p} \sqcup DE$. *c* therefore $DL \sqcup GL$. *d* but $DLq \sqcup GLq$. More- *c* $13. 10.$ over DL *e* $\sqcup \sqrt{DLq - GLq}$. wherefore DG *d* *sch. 12.* *f* is a fifth residual. Which was to be demonstrated. *10.*

P R O P. CIII.

e *lem. 97.*
10.
f $5. def. 85.$
10.

See the Scheme prop. 100.

The square of a line AB ($AC - BC$) making with a medial space the whole space medial, applied to a rational line DE , makes the breadth DG a sixth residual line.

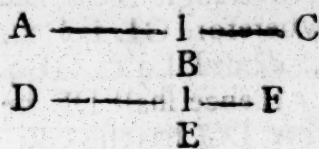
As above, DK and GK are $\mu \alpha$; *a* wherefore DL and GL are $\dot{p} \sqcup DE$. also $DK b \sqcup GK$. *c* whence $DL \sqcup GL$. *d* therefore DG is a residual. *b* And whereas $ACq \sqcup BCq$. and so $DL \sqcup \sqrt{DLq - GLq}$, *e* therefore DG shall be a sixth residual. Which was to be demonstrated. *c* $10. 10.$

a $23. 10.$
b *hyp. and*
lem. 97. 10.
c $10. 10.$
d $74. 10.$
e $6. def. 85.$
10.

R.

P R O P.

P R O P. CIV.



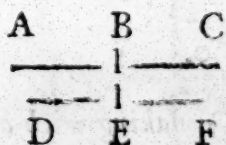
A right line DE commensurable in length to a residual AB (AC - BC) is it self also a residual, and of the same order.

Lemma.

Let AB. DE :: AC. DF. and AB $\sqsubset$ DE.

- I say AB + BC $\sqsubset$ DF + EF. For AC. BC a :: DF. EF. therefore by addition AC + BC. BC :: DF + EF. EF. therefore by inversion AC + BC. DF + BC $\sqsubset$ DF + EF. Which was to be dem.
- a lem. 66. EF :: BC. EF. a but BC $\sqsubset$ EF. b therefore AC + BC $\sqsubset$ DF + EF. Which was to be dem.
- b 10. 10. a Make AB. DE :: AC. DF. b therefore AC + BC $\sqsubset$ DF + EF. therefore seeing AC + BC c is a binomial, d DF + EF shall be a binomial too, and of the same order. e wherefore DF - EF is a residual of the same order with AC - BC. Which was to be demonstrated.
- c hyp. d 67. 10. e by def. 85.
- 10.

P R O P. CV.



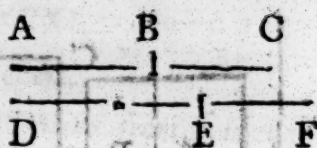
A right line DE commensurable to a medial residual line AB (AC - BC) is it self a medial residual, and of the same order

- Again a make AB. DE :: AC. DF. b whence AC + BC $\sqsubset$ DF + EF. c therefore DF + EF is a bimedral of the same order with AC + BC, d and consequently DF - EF shall be a medial residual of the same order with AC - BC. Which was to be demonstrated.
- a 12. 6. b lem. 103. c 68. 10. d 75. and 76. 10.

P R O P.

PROP. CVI.

A right line DE commensurable to a Minor line AB (AC - BC) is it self also a Minor line.



Make AB. DE :: AC.

DF. a then is AC + BC = DF + EF. but AC + BC a lem. 103. b is a Major line; c therefore DF + EF is also a 10. Major line; d and consequently DF - EF is a b hyp. Minor line. Which was to be demonstrated. c 69. 10. d 77. 10.

PROP. CVII.

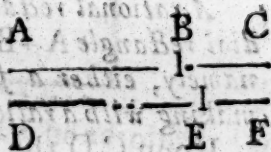
A right line DE commensurable to a line AB (AC - BC) which makes with a rational space the whole space medial, is it self also a line making with a rational space the whole space medial.



For, accordingly as in the former, we may shew DF + EF to contain in power $\rho\nu$ and $\mu\nu$. a whence a 78. 10. DF - EF is a line making, &c.

PROP. CVIII.

A right line DE commensurable to a line AB (AC - BC) which with a medial space makes the whole space medial, is it self a line making with a medial space the whole space medial.

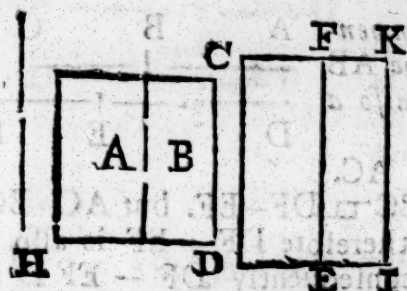


For according to the preceding DF + EF shall contain in power 2 $\mu\nu$. a therefore DF - EF shall a 79. 10. be, as in the Prop.

R 2

PROP.

P R O P. CIX.



A medial rectangle B being taken from a rational rectangle A+B, the right line H which containeth in power the space remaining A, is one of these two irrational lines, viz. either a residual line, or a Minor line.

Upon CD ρ make the rectangles CI=A+B, and FI=B. whence CE=A=B. wherefore because CI b is ρv . c therefore CK is $\rho \sqsubset$ CD. but being FI b is μv , d shall FK be $\rho \sqsubset$ CD. e whence CK $\sqsubset$ FK. f therefore CF is a residual line. Wherefore if CK be $\sqsubset \sqrt{CKq} - FKq$, g then CF shall be a first residual. h therefore $\sqrt{CE}$ (H) is a residual line. But if CK $\sqsubset \sqrt{CKq} - FKq$, k then CF shall be a fifth residual; and consequently H ($\sqrt{CE}$) shall be a Minor line. Which was to be dem.

a 3. ax. 1.
b hyp. and
constr.
c 21. 10.
d 23. 10.
e 13. 10.
f 74. 0.
g 1. def. 85.
10.
h 92. 10.
k 4. def. 85.
10.
l 95. 10.

P R O P. CX.

See the preceding Scheme.

A rational rectangle B being taken away from a medial rectangle A+B, other two irrational lines are made, namely, either a first medial residual line, or a line

a hyp. and making with a rational space the whole space medial.
constr. Upon CD the propounded ρ make the rectangles CI=A+B, and FI=B. a whence CE=A=B. Therefore because CI b is μv ; c shall CK be $\rho \sqsubset$ CD. but because FI b is ρv . d thence FK $\rho \sqsubset$ CD. e whence CK $\sqsubset$ FK. f therefore CF is a residual, g and that a second. If CK $\sqsubset \sqrt{CKq} - FKq$, h then H ($\sqrt{CE}$) is a first medial residual. But if CK $\sqsubset \sqrt{CKq} - FKq$, k then shall CF be a fifth residual; and l consequently H ($\sqrt{CE}$) shall be a line making μv with ρv . Which was to be dem.

P R O P.

PRO P. CXI.

See the same Scheme.

A medial space B being taken away from a medial space A+B, which is incommensurable to the whole A+B, the other two irrational lines are made, viz. either a second medial residual line, or a line making with a medial space the whole space medial.

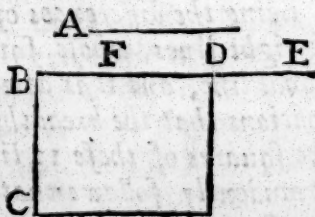
Upon CD ρ make the rectangles CI=A+B, and FI=B. a wherefore CE=A=Hq. Because therefore CI is $\mu\nu$, b thence CK is $\rho \sqsubset$ CD. and in like manner FK $\rho \sqsubset$ CD. Likewise because CI $\sqsubset$ FI, d therefore CK $\sqsubset$ FK. e wherefore CF is a residual, f namely a third. If CK $\sqsubset \sqrt{CKq}$ - FKq, g; whence H ($\sqrt{CE}$) shall be a second medial residual, but if CK $\sqsubset \sqrt{CKq}$ - FKq h then shall CF be a sixth residual. k wherefore A shall be a line making $\mu\nu$ with μ . Which was to be demonstrated.

a 3. ax. 1.
b 23. 10.
c hyp.
d 10. 10.
e 74. 10.
f 3. def. 85.
g 94. 10.
h 6. def. 85.
i 10.
k 97. 10.

PRO P. CXII.

A residual line A is not the same with a binomial line.

Upon BC propounded ρ make the rectangle CD=Aq. Therefore seeing A is a residual, a BD shall be a first residual, to which let



DE be the line congruent or that may be adjoined. b wherefore BE, DE, are $\rho \sqsubset$. c and BE $\sqsubset$ BC. If you conceive A to be a binomial, then BD is a first binomial, whose names let be BF, FD; and let BF be $\sqsubset$ FD. d therefore BF, FD are $\rho \sqsubset$; and BF $\sqsubset$ BC. therefore since BC $\sqsubset$ BE, f shall BE be $\sqsubset$ BF. g and thence BE $\sqsubset$ FE. h therefore FE is ρ . Likewise because BE $\sqsubset$ DE, k shall FE be $\sqsubset$ DE. l wherefore FD is a residual, and so FD is ρ . but it was shewn ρ . which are repugnant, therefore A is falsly conceived to be a binomial. Which was to be demonstrated.

a 98. 10.
b 74. 10.
c 1. def. 85.
d 37. 10.
e 1. def. 48.
f 10.
g 12. 10.
h cor. 16.
i 10.
j sch. 12.
k 14. 10.
l 74. 10.

R 3

The 174. 10.

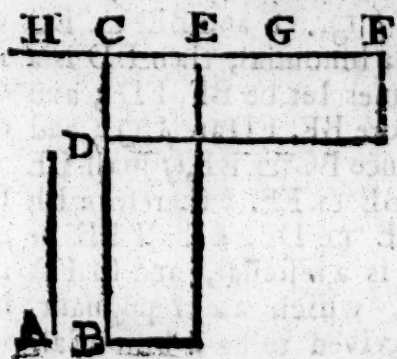
The tenth Book of

The names of the 13 irrational lines differing one from another.

1. A Medial line.
2. A binomial line ; of which there are six species.
3. A first bimedial line.
4. A second bimedial.
5. A Major line.
6. A line containing in power a rational superficies, and a medial superficies.
7. A line containing in power two medial superficies.
8. A residual line ; of which there are also six kinds.
9. A first medial residual line.
10. A second medial residual line.
11. A Minor line.
12. A line making with a rational superficies the whole superficies medial.
13. A line making with a medial superficies the whole superficies medial.

Being the differences of breadths do argue differences of right lines, whose squares are applied to some rational line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these 13 lines do differ one from another, it evidently follows that these 13 lines do also differ one from another.

P R O P. CXIII.

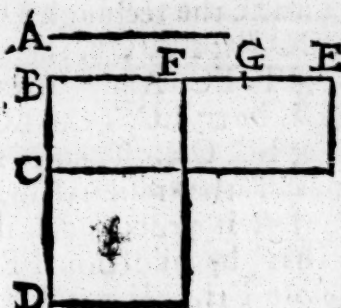


The square of a rational line A applied to a binomial BC (BD + DC) makes the breadth EC a residual line, whose names EH, CH, are commensurable to the names BD, DC, of the binomial line, and in the same proportion (EH. BD :: CH.

CH. DC:) and moreover, the residual line EC which is made, is of the same order with BC the binomial.

Upon DC the less name *a* make the rectangle DF a cor. 16. 6.
 = Aq = BE. whence BC. CD *b* :: FC. CE. there b 14. 6.
 fore by division, BD. DC :: FE. EC, And whereas
 BD *c* $\square$ DC, *d* thence FE shall be $\square$ EC, Take EG *c* hyp.
 = EC, and make FG. GE :: EC. CH. Then EH, *d* 14. 5.
 and CH shall be the names of the residual EC,
 whereunto all is agreeable that is propounded in
 the theoreme. For being that by addition FE.
 GE (EC) :: EH. CH, therefore FH. EH *e* :: EH. *e* 12. 5.
 CH *f* :: FE. EC *f* :: BD. DC. wherefore since BD *g* *f* before.
 $\sqsupset$ DC, *h* thence shall EH be $\sqsupset$ CH, *h* and FH *g* hyp.
 $\sqsupset$ EHq. Therefore because FHq. EHq *k* :: FH. *h* 10. 10.
 CH. *l* shall FH be $\sqsupset$ CH, *l* and so FC $\sqsupset$ CH. *k* cor. 20. 6.
 Moreover CD *g* is $\dot{p}$, and DF (Aq) *g* is $\dot{p}$. *m* there- *l* 16. 10.
 fore FC is $\dot{p}$, $\sqsupset$ CD. whence also CH is $\dot{p}$ $\sqsupset$ CD. *m* 21. 10.
n therefore EH, CH are $\dot{p}$ and $\sqsupset$, as before. *o* *n* sch 12.
 therefore EC is a residual line, to which CH may *o* 10.
 be joined. Furthermore EH. CH *f* :: BD. DC. and *o* 74. 10.
 so by inversion EH. BD :: CH. DC. whence because
 CH *f* $\sqsupset$ DC, *p* shall EH be $\sqsupset$ BD. But suppose *p* 10. 10.
 BD $\sqsupset$ $\sqrt{BDq - DCq}$. *q* then shall EH be $\sqsupset$ *q* 15. 10.
 $\sqrt{EHq - CHq}$. Also if BD $\sqsupset$ $\dot{p}$ propounded, *r* 12. 10.
 then shall EH be $\sqsupset$ to the same $\dot{p}$. *s* that is, if *s* 1. def. 48.
 BC be a first binomial, *t* EC shall be a first residual. *t* 10.
 In like manner, if DC be to the $\sqsupset$ propounded, *t* 1. def. 85.
 $\dot{p}$, *r* then is CH $\sqsupset$ to the same $\dot{p}$. *u* that is, if BC *u* 10.
 be a second binomial, *x* EC shall be a second re- *u* 2. def. 48.
 sidual: and if this be a third binom. then that *u* 10.
 shall be a third residual, &c. But if BD be $\sqsupset$ $\sqrt{BDq - DCq}$, *x* 2 def. 85.
 BDq - DCq, *y* then shall EH be $\sqsupset$ $\sqrt{EHq - CHq}$. *y* 10.
 CHq. therefore if BC be a 4, 5, or 6 binomial, *y* 15. 10.
 EG shall be likewise a 4, 5, or 6 residual. Which
 was to be demonstrated.

P R O P. CXIV.



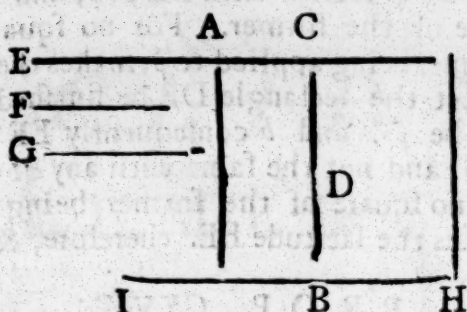
The square of a rational line A applied to a residual line BC ($BD - CD$) makes the breadth BE a binomial; whose names BE, GE are commensurable to the names BD, BC of the residual line BC, and in the same proportion.

and moreover, the binomial line which is made (BE) is of the same order with the residual line (BC.)

- a *cor.* 16. 6. a Make the rectangle $DF = Aq$. and $BF \cdot FE$ b
 b 12. 6. $:: EG \cdot GF$. whence for that $DF = Aq = CE$, c
 c 14. 6. therefore $BD \cdot BC :: BE \cdot BF$. therefore by conver-
 d 19. 5. sion of proportion $BD \cdot CD :: BE \cdot FE :: EG \cdot GF$
 e *hyp.* $:: d \cdot BG \cdot EG$. but BD e $\perp$ CD . f therefore BG
 f 10. 10. $\perp$ GE . therefore because $BGq \cdot GEq g :: BG \cdot GF$.
 g *cor.* 20. 6. h shall BG be $\perp$ GF . k and so $BG \perp BF$. more-
 h 10. 10. over BD e is ρ , and the rectangle $DF (Aq)$ e is $\rho \vee l$
 k *cor.* 16. therefore BF is $\rho \perp BD$. m therefore also BG is
 10. $\rho \perp BD$. n therefore BG, GE are $\rho \perp$. o where-
 l 21. 10. fore BE is a binomial. Lastly, because $BD \cdot CD$
 m 12. 10. $:: BG \cdot GE$. and inversely $BD \cdot BG :: CD \cdot GE$. and
 n *sch.* 12. $BD \perp BG$. p thence shall CD be $\perp$ GE . there-
 10. fore if CB be a first residual, BE shall be a first
 o 37. 10. binomial, &c. as in the *prec.* therefore, &c.
 p 10. 10.

PROP.

P R O P. CXV.



If a space AB be contained under a residual line AC ($CE - AE$) and a binomial CB , whose names CD , DB are commensurable to the names CE , AE , of the residual line, and in the same proportion ($CE. AE :: CD. DB$) then the right line F which containeth in power that space AB , is irrational.

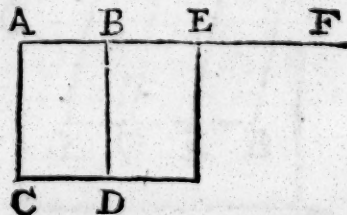
Let G be ρ . and make the rectangle $CH = Gq$; athen shall BH ($HI - IB$) be a residual line, and HI $a \sqsupset CD$ $b \sqsupset CE$, a and BI $\sqsupset DB$, a and HI . a 113. 10
 $BI :: CD. DB$ $b :: CE. EA$. therefore by inversion b hyp.
 $HI. CE :: BI. EA$. c therefore $BH. AC :: HI. CE$ c 19. 5.
 $:: BI. EA$. wherefore since $d HI \sqsupset CE$, e thence d 12. 10.
 $BH \sqsupset AC$. f therefore the rectangle $HC \sqsupset BA$. e 10. 10.
 But HC (Gq) b is ρv . g therefore BA (Fq) is ρv . f 1. 6. and
 and consequently F is ρ . Which was to be dem. g sch. 12.
10.

Coroll.

Hereby it appears that a rational superficies may be contained under two irrational right lines.

P R O P. CXVI.

Of a medial line AB are produced infinite irrational lines BE , EF , &c. whereof none is of the same kind with any of the precedent.

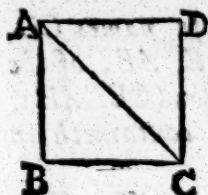


Let AC be propounded ρ . and AD a rectangle con-

contained under AC, AB. *a* therefore AD is $\dot{p}v$. Take BE = $\sqrt{AD}$. *b* then BE is $\dot{p}$, and the same with none of the former. For no square of any of the former being applied to $\dot{p}$, makes the breadth medial. Let the rectangle DE be finished, *a* then DE shall be $\dot{p}v$. and *b* consequently EF ($\sqrt{DE}$) shall be $\dot{p}$, and not the same with any of the former. for no square of the former being applied to $\dot{p}$, makes the latitude BE. therefore, &c.

a lem. 38.
10.
b 11. 10.

P R O P. CXVII.



a 47. 1.
bcor. 24. 8.
c 9. 10.

Let it be required to shew that in square figures BD, the diameter AC is incommensurable in length to the side AB.

For ACq. ABq $a :: 2. 1 b ::$ not Q. Q. *c* therefore AC $\nparallel$ AB. Which was to be dem. This theo-

reme was of great note with the ancient Philosophers; so that he that understood it not was esteemed by Plato undeserving the name of a man, but rather to be reckoned among brutes.

The End of the tenth Book.

THE

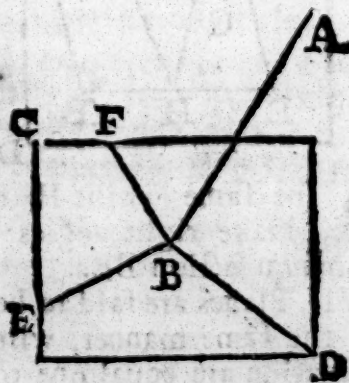
The ELEVENTH BOOK
OF
EUCLIDE'S
ELEMENTS.

Definitions.

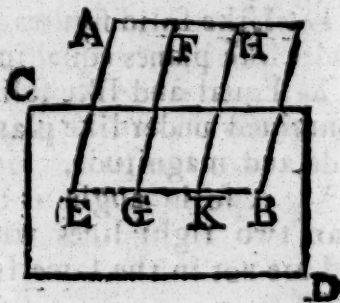
I. **A** Solid is that which hath length, breadth and thickness.

II. The term, or extreme of a solid is a Superficies.

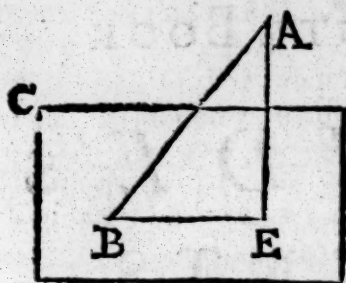
III. A right line AB is perpendicular to a Plane CD, when it makes right angles ABD, ABE, ABE, with all the right lines BD, BE, BF, that touch it, and are drawn in the said Plane.



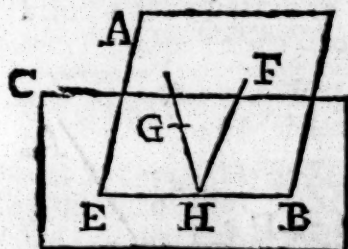
IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common section of the two Planes EB, and making right angles therewith, do also make right angles with the other Plane CD.



V. The



V. The inclination of a right line AB to a Plane CD, is when a perpendicular AE is drawn from A the highest point, of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the plane CD, to the end B of the said line AB which is in the same plane, whereby the angle, is acute ABE which is contained under the insisting line AB, and the line drawn in the plane EB.



VI. The inclination of a plane AB to a plane CD, is an acute angle FGH contained under the right lines FH, GH which being drawn in either of the planes AB, CD to the same point H of the common section BE, make right angles FHB, GHB, with the common section BE.

VII. Planes are said to be inclined to other planes in the same manner, when the said angles of inclination are equal one to another.

VIII. Parallel planes are those which being prolonged never meet.

IX. Like solid figures are such as are contained under like planes equal in number.

X. Equal and like solid figures are such as are contained under like planes equal both in multitude and magnitude.

XI. A solid angle is the inclination of more than two right lines which touch one another, and are not in the same superficies,

Or thus ;

A solid angle is that which is contained under more than two plane angles not being in the same superficies, but consisting all at one point.

XII. A Pyramide is a solid figure comprehended under divers planes set upon one plane (which is the base of the pyramide,) and gathered together to one point.

XIII. A Prisme is a solid figure contained under planes, whereof the two opposite are equal, like, and parallel ; but the others are parallelograms.

XIV. A Sphere is a solid figure made when the diameter of a semicircle abiding unmoved, the semicircle is turned round about, till it return to the same place from whence it began to be moved.

Coroll.

Hence, all the rays drawn from the center to the superficies of a sphere, are equal amongst themselves.

XV. The Axis of a sphere, is that fixed right line, about which the semicircle is moved.

XVI. The Center of a sphere, is the same point with the center of the semicircle.

XVII. The Diameter of a sphere, is a right line drawn through the center, and terminated on either side in the superficies of the sphere.

XVIII. A Cone is a figure made, when one side of a rectangled triangle (*viz.* one of those that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it first moved. And if the fixed right line be equal to the other which containeth the right angle, then the Cone is a rectangled Cone ; but if it be less, it is an obtuse-angled Cone ; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX.

XX. The Base of a Cone is the circle, which is described by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the sides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the same place, where it began to move.

XXII. The Axis of a Cylinder is that quiescent right line, about which the parallelogram is turned.

XXIII. And the Base of a Cylinder are the circles which are described by the two opposite sides in their motion.

XXIV. Like Cones and Cylinders, are they, both whose Axes and Diameters of their Bases are proportional.

XXV. A Cube is a solid figure contained under six equal squares.

XXVI. A Tetraedron is a solid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a solid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a solid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icosaedron is a solid figure contained under twenty equal and equilateral triangles.

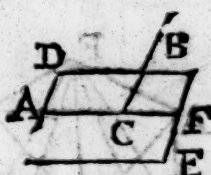
XXX. A Parellelipedon is a solid figure contained under six quadrilateral figures, whereof those which are opposite are parallel.

XXXI. A solid figure is said to be inscribed in a solid figure, when all the angles of the figure inscribed are comprehended either within the angles, or in the sides, or in the planes of the figure wherein it is inscribed.

XXXII. Likewise a solid figure is then said to be circumscribed about a solid figure, when either the angles, or sides, or planes of the circumscribed figure touch all the angles of the figure which it contains.

PROPOSITION. I.

One part AC of a right line cannot be in a plane superficies, and another part CB elevated upward.



Produce AC in the plane directly to F. If you conceive CB to be drawn strait from AC, then two right lines AB, AF, have one common segment AC. *a Which is impossible.*

a 10. ax. 1.

PROP. II.

If two right lines AB, CD, cut one another, they are in the same plane: And every triangle DEB is in one and the same plane.

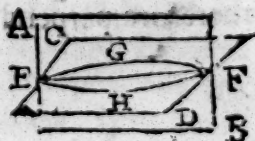


For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right line ED is in a plane, and the other part elevated upwards. *a Which is absurd.* Therefore the triangle EDB is in one and the same plane; and so also are the right lines ED, EB; *a wherefore the whole lines AB, DC, are in one plane. Which was to be demonstrated.*

a 1. 11.

PROP. III.

If two planes AB, CD, cut one the other, their common section EF is a right line.



If EF the common section be not a right line, a then in

the plane AB draw the right line EGF. *a* and in the plane CD the right line EHF. therefore two right lines EGF, EHF include a superficies. *b Which is absurd.*

a 1. post. 1.

b 14. ax. 1.

PROP.

XX. The Base of a Cone is the circle, which is described by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the sides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the same place, where it began to move.

XXII. The Axis of a Cylinder is that quiescent right line, about which the parallelogram is turned.

XXIII. And the Base of a Cylinder are the circles which are described by the two opposite sides in their motion.

XXIV. Like Cones and Cylinders, are they, both whose Axes and Diameters of their Bases are proportional.

XXV. A Cube is a solid figure contained under six equal squares.

XXVI. A Tetraedron is a solid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a solid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a solid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icofaedron is a solid figure contained under twenty equal and equilateral triangles.

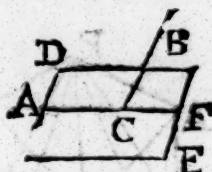
XXX. A Parellelipedon is a solid figure contained under six quadrilateral figures, whereof those which are opposite are parallel.

XXXI. A solid figure is said to be inscribed in a solid figure, when all the angles of the figure inscribed are comprehended either within the angles, or in the sides, or in the planes of the figure wherein it is inscribed.

XXXII. Likewise a solid figure is then said to be circumscribed about a solid figure, when either the angles, or sides, or planes of the circumscribed figure touch all the angles of the figure which it contains.

PROPOSITION. I.

One part *AC* of a right line cannot be in a plane superficies, and another part *CB* elevated upward.



Produce *AC* in the plane directly to *F*. If you conceive *CB* to be drawn strait from *AC*, then two right lines *AB*, *AF*, have one common segment *AC*. *a Which is impossible.*

a 10. ax. 1.

PROP. II.

If two right lines *AB*, *CD*, cut one another, they are in the same plane: And every triangle *DEB* is in one and the same plane.

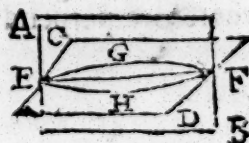


For imagine *EFG*, part of the triangle *DEB*, to be in one plane, and the part *FDGB* to be in another, then *EF* part of the right line *ED* is in a plane, and the other part elevated upwards. *a Which is absurd.* Therefore the triangle *EDB* is in one and the same plane; and so also are the right lines *ED*, *EB*; *a* wherefore the whole lines *AB*, *DC*, are in one plane. *Which was to be demonstrated.*

a 1. 11.

PROP. III.

If two planes *AB*, *CD*, cut one the other, their common section *EF* is a right line.



If *EF* the common section be not a right line, *a* then in

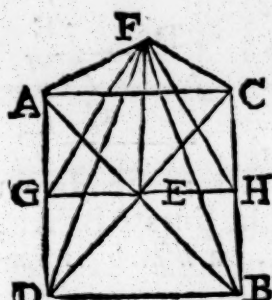
the plane *AB* draw the right line *EGF*. *a* and in the plane *CD* the right line *EHF*. therefore two right lines *EGF*, *EHF* include a superficies. *b Which is absurd.*

a 1. post. 1.

b 14. ax. 1.

PROP.

P R O P. IV.



If a right line EF be at right angles erected upon two lines AB, CD, cutting one the other, at the common section E; it shall also be at right angles to the plane ACBD drawn by the said lines.

a constr.

b 15. 1.

c 4. 1.

d sch. 34. 1.

e 29. 1.

f constr.

g 26. 1.

h 4. 1.

k 8. 1.

l 4. 1.

m 8. 1.

n 10. def. 1.

o 3. def. 11.

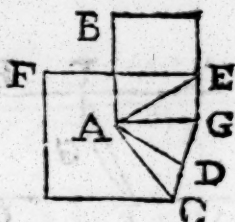
Take EA, EC, EB, ED, equal one to the other, and join the right lines AC, CB, BD, AD. draw any right line GH through E, and join FA, FC, FD, FB, FG, FH. Because AE is $a = EB$, and DE $a = EC$, and the angle AED $b = CEB$, c therefore AD is $= CB$, c and likewise AC $= DB$. d therefore AD is parallel to CB, d and AC to BD. e wherefore the angle GAE $= EBH$, and the angle AGE $= EHB$. But also AE $f = EB$. g therefore GE $= EH$, g and AG $= BH$. whence by reason of the right angles, by the hyp. and so equal, at E, h the bases FA, FC, FB, FD, are equal. Therefore the triangle ADF, FBC, are equilateral one to another. k and thence the angle DAF $= BCF$. Therefore in the triangles AGF, FBH, the sides FG, FH l are equal; and so by consequence the triangles FEG and FEH are mutually equilateral. m therefore the angles FEG, FEH are equal, and n so right angles. In like manner, FE makes right angles with all the lines drawn through E in the plane AD, BC, o and is therefore perpendicular to the said plane.

P R O P.

PROP. V.

If a right line AB be erected perpendicular to three right lines AC, AD, AE, touching one the other at the common section, those three lines are in the same plane.

For AC, AD, *a* are in one plane FC; *a* and AD, AE, are in one plane BE. which if you conceive to be several planes, then let their intersection *b* be the right line AG; therefore because BA by the Hyp. is perpendicular to the right lines AC, AD. *c* and so to the plane FC, *d* it is also perpendicular to the right line AG. therefore (since *a* that AB is in the same plane with AC, AE) the angles BAG, BAE, are right angles, and consequently equal, the part and the whole. Which is absurd.



a 2. 11.

b 3. 11.

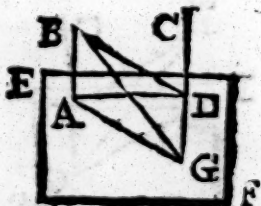
c 4. 11.

d 3. def. 11.

PROP. VI.

If two right lines AB, DC, be erected perpendicular to one and the same plane EF, those right lines AB, DC are parallel one to the other.

Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and join BD, BG, AG. Being in the triangles BAD, ADG, the angles DAB, ADG *a* are right angles, and AB *b* = DG, and AD is common, *c* therefore BD is = AG. whence in the triangles AGB, BGD, equilateral one to the other, the angle BAG is *d* = BDG; of which being BAG is a right angle, BDG shall be so also. but the angle GDC is supposed right, therefore the right line GD is perpendicular to the three lines DA, DE, CD. *e* which are therefore in the same plane *f* wherein AB is. Wherefore since AB and CD are in the same plane, and the internal angles BAD, CDA, are right angles, *g* AB and CD shall be parallels. Which was to be demonstrated.



a hyp.

b constr.

c 4. 1.

d 8. 1.

e 5. 11.

f 2. 11.

g 28. 1.

S

PROP.

P R O P. VII.

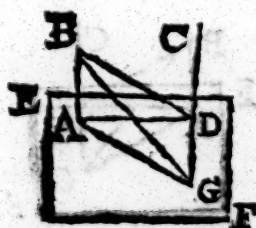


If there be two parallel right lines AB, CD , and any points E, F , be taken in both of them, the line EF which is joined at these points, is in the same plane with the parallels AB, CD .

Let the plane in which AB, CD , are, be cut by another plane at the points E, F . then if EF is not in the plane $ABCD$, it shall not be the common section. Therefore let EGF be the common section; which a then is a right line. therefore two right lines EF, EGF , include a superficies. b Which is absurd.

a 3. II.
 b 14. ax.
10.

P R O P. VIII.



If there be two parallel right lines AB, CD , whereof one AB , is perpendicular to a plane EF . then the other CD shall be perpendicular to the same plane EF .

The preparation and demonstration of the sixth of this

Book being transferr'd hither; the angles GDA , and GDB are right angles: a therefore GD is perpendicular to the plane, wherein are AD, DB (b in which also AB, CD , are.) c therefore GD is perpendicular to CD . but the angle CDA is also d a right angle. e therefore CD is perpendicular to the plane EF . Which was to be demonstrated.

a 4. II.

b 7. II.

c 3. def. II.

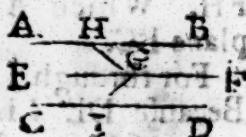
d 29. 1.

e 4. II.

P R O P.

PROP. IX.

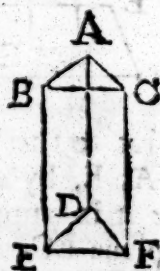
Right lines (AB, CD) which are parallel to the same right line EF , but not in the same plane with it, are also parallel one to the other.



In the plane of the parallels AB, EF , draw HG perpendicular to EF ; also in the plane of the parallels EF, CD , draw IG perpendicular to EF . *a* therefore EG is perpendicular *a* 4. 11. to the plane wherein HG, GI are; and AH, CI are *b* 8. 11. perpendicular to the same plane. *c* therefore AH *c* 6. 11. and CI are parallels. Which was to be demonstrated.

PROP. X.

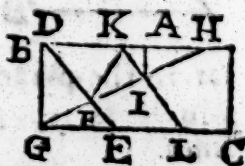
If two right lines AB, AC , touching one another be parallel to two other right lines ED, DF , touching one another, and not being in the same plane, those right lines contain equal angles, BAC, EDF .



Let AB, AC, DE, DF , be equal one to the other, and draw AD, BC, EF, BE, CF . Being AB, DE , *a* are parallels and equal, *b* also BE, AD , are parallels *constr.* and equal. In like manner CF, AD , are parallels *b* 33. 1. and equal; *c* therefore also BE, FC , are parallels *c* 2. *ax.* 1. and equal. *d* Therefore BC, EF are equal. Where- *and* 30. 1. fore since the triangles BAC, EDF , are of equal *d* 33. 1. sides one to the other, the angles BAC, EDF *e* 8. 1. shall be equal. Which was to be demonstrated.

PROP. XI.

From a point given on high A to draw a right line AI perpendicular to a plane below BC .



In the plane BC draw any line DE ; to which from the point A draw the perpendicular AF , and *b* like- *a* 12. 1. *b* 11. 1. wise

a 12. 1.

c 31. 1.

d constr.

e 4. 11.

f 8. 11.

g 3. def. 11.

h constr.

l 4. 11.

wise FH in the plane BC cutting the said line DE at F; *a* then let fall AI shall be perpendicular to FH. Which AI shall be perpendicular to the plane BC.

For through I *c* let KIL be drawn parallel to DE. Because DE *d* is perpendicular to AF, and FH, *e* therefore DE shall be perpendicular to the plane IFA. and so also KL *f* is perpendicular to the same plane. *g* therefore the angle KIA is a right angle. but the angle AIF is also *b* a right angle. *l* therefore AI is perpendicular to the plane BC. Which was to be done.

P R O P. XII.



a 11. 11.

b 31. 1.

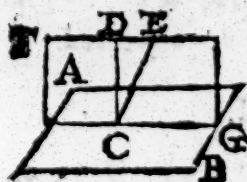
c 8. 11.

In a plane given BC, at a point given therein A, to erect a perpendicular line AF.

From some point without the plane, D, *a* draw DE perpendicular to the said plane BC. and joining the points A, E, by a line AE, *b* draw AF parallel to DE. *c* it is apparent that AF is perpendicular to the plane BC. Which was to be done.

This and the preceding problem are practically performed by applying two Squares to the point given; as appears by 4. 11.

P R O P. XIII.



a 6. 11.

At a point given C in a plane given AB, two right lines CD, CE, cannot be erected perpendicular on the same side.

For both CD, and CE *a* should then be perpendicular to the plane AB. and consequently parallels; which is repugnant to the definition of parallel lines.

P R O P.

PROP. XIV.

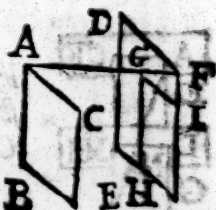
Planes CD , FE , to which the same right line AB is perpendicular, are parallel.

If you deny this; then let the planes CD , EF meet, so that their common section be the right line GH . in which take any point I , draw to it the right lines IA , IB , in the said planes. whereby in the triangle IAB , two angles IAB , IBA *a* are right angles. *b* Which *a* hyp. and *b* 3. def. 11. is absurd. *b* 17. 1.



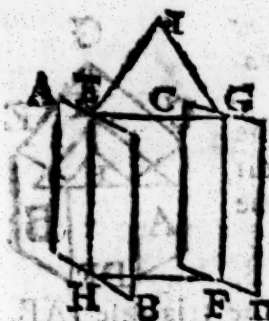
PROP. XV.

If two right lines AB , AC , touching one the other, be parallel to two other right lines DE , DF , touching one the other, and not being in the same plane with them, the planes BAC , EDF , drawn by those right lines are parallel one to the other.



From A *a* draw AG perpendicular to the plane *a* 11. 11. EF . *b* and let GH , GI be parallel to DE , DF . *b* 31. 1. *c* these also shall be parallel to AB , AC . Therefore *c* 30. 1. since the angles IGA , HGA , *d* are right angles, *d* 3. def. 11. also CAG , BAG , *e* shall be right angles. *e* 29. 1. *f* therefore GA is perpendicular to the plane BC ; but the *f* 4. 11. same is perpendicular to the plane EF . *g* therefore *g* constr. the planes BC , EF , are parallel. *h* 14. 11. Which was to be demonstrated.

P R O P. XVI.



If two parallel planes AB, CD , be cut by some other plane $HEIGP$, their common sections EH, GF are parallel one to the other.

For if they be conceived to be otherwise; being in the same plane that cuts them, they will meet some where, if produced; suppose in I ; wherefore since the whole lines HEI, FGI are in the planes AD, CD , being produced, the planes also shall meet. *contrary to the Hyp.*

a 1. 11.

P R O P. XVII.



If two right lines ALB, CMD , be cut by parallel planes EF, GH, IK ; they shall be cut porportionally, ($AL. LB :: CM. MD.$)

Let the right lines AC, BD , be drawn in the planes EF, IK ; as also AD passing through the plane GH in the point N . and join NL, LM . the planes of the triangles ADC, ADB , make the sections BD, LN , and AC, NM a parallels. Therefore $AL. LB :: AN. ND$ *b 6.* $CM. MD.$ Which was to be demonstrated.

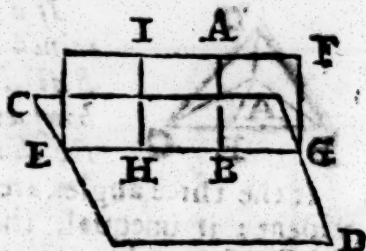
a 16. 11.

b 2. 6.

P R O P.

P R O P. XVIII.

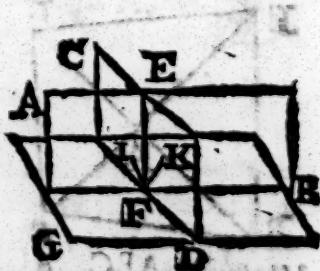
If a right line AB be perpendicular to some plane CD , all the planes extended by that right line AB (EF , &c.) shall be perpendicular to the same plane CD .



Let there be some plane EF drawn by AB , making the section EG with the plane CD ; from some point whereof H , draw HI parallel to AB in the plane EF ; then shall HI be perpendicular to the plane CD , and so likewise any other lines, that are perpendicular to EG . therefore the plane EF is perpendicular to the plane CD ; and by the same reason any other planes drawn by AB shall be perpendicular to EF . Which was to be demonstrated.

P R O P. XIX.

If two planes AB, CD , cutting one the other, be perpendicular to some plane GH , their line of common section EF shall be perpendicular to the same plane (GH .)



Because the planes AB, CD , are taken perpendicular to the plane GH , it appears by 4. def. 11. that out of the point F there may be drawn in both planes AB, CD , a perpendicular to the plane GH , which shall be a but one; and therefore the common section of the said planes. Which was to be demonstrated.

P R O P. XX.

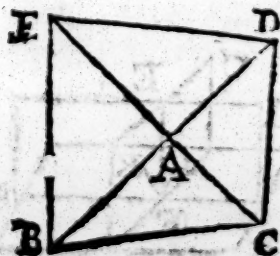


If a solid angle $ABCD$ be contained under three plane angles, BAD, DAC, BAC , any two of them howsoever taken are greater than the third.

If the three angles are equal, the assertion is evident; if unequal, then let the the greatest be BAC ; from whence a take away $BAE = BAD$, and make $AD = AE$; and also draw BEC, BD, DC .

Because the side BA is common, and $AD = AE$; and the angle $BAE = BAD$, c thence is $BE = BD$. but $BD + DC$ is $d < BC$. e therefore $DC < EC$. Wherefore since $AD = AE$, and the side AC is common, and $DC < EC$, f the angle CAD shall be $< EAC$. g therefore the angle $BAD + CAD < BAC$. Which was to be demonstrated.

P R O P. XXI.



Every solid angle A is contained under less angles than four plane right angles.

For let a plane any-wise cutting the sides of the solid angle A make a many-sided figure $BCDE$, and as many triangles ABC, ACD, ADE, AEB . I denote all the angles of the polygone by X ; and I term the sum of the angles at the bases of the triangles Y . wherefore $X + 4$ right angles $a = Y + A$. but being that (of all the angles at B) b the angle $ABE + ABC$ is $= CBE$, and the same is true also of the angles at C , at D , and at E , c it is manifest that Y is $= X$. and consequently A shall be < 4 right gles. Which was to be demonstrated.

P R O P.

P R O P. XXII.

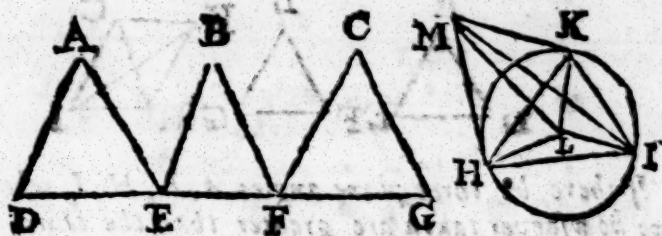


If there be three plane angles A, B, HCI , whereof two howsoever taken are greater than the third, and the right lines which contain them be equal AD, AE, FB , &c. then of the right lines DE, FG, HI , coupling those equal right lines together, it is possible to make a triangle.

A triangle may be made of them, if any two be greater than the third: but they are so. For make the angle $HCK = B$, and $CK = CH$, and draw HK, IK . thence $KH = FG$ and because the angles KCI but $KI \neq HI + KH$ (FG.) therefore $DE \neq HI + FG$. By the like argument any two may be proved greater than the third; and consequently it is possible to make a triangle of them. Which was to be demonstrated.

P R O P.

P R O P. XXIII.



To make a solid angle $MHIK$ of three plane angles A, B, C , whereof two howsoever taken are greater than the third. * But it is necessary that those three angles be less than four right angles.

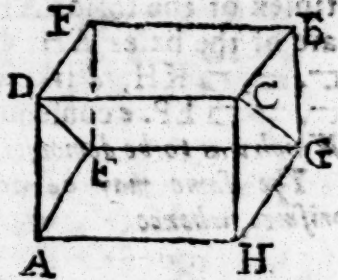
Make AD, AE, BE, CF, FG , equal one to the other; and of the subtended lines DE, EF, FG (that is, of the equal lines HI, IK, KH) *a* make the triangle HKI ; about which *b* describe the circle $LHKI$. * But because AD is $\perp HL$, *c* let ADq be $= HLq + LMq$. *d* and let LM be perpendicular to the place of the circle HKI . and draw HM, KM, IM . wherefore since the angle HLM *e* is a right angle, *f* thence is $MHq = HLq + LMq$ *g* $= ADq$. therefore $MH = AD$. By the same reason MK, MI, AD (that is $AE, EB, &c.$) are equal; therefore since $HM = AD$, and $MI = AE$, and $DE = HI$, *k* the angle A shall be $= HMI$, *k* as likewise the angle $IMK = B$, *k* and the angle $HMK = C$, wherefore a solid angle is made at M of the three given plane angles. Which was to be done. AD is assumed to be $\perp HL$. But this is manifest. For if AD be $=$ or $\perp HL$, then is the angle $A = m$ or $\perp HLI$. In like manner shall B be equal or $\perp HLK$, and $C =$ or $\perp KLI$, wherefore $A + B + C$ * shall either equal or exceed four right angles. contrary to the Hypoth. therefore rather let AD be $\perp HL$. Which was to be dem.

l constr.
and 8. 1.
m 21. 1.
* 4 cor. 13.
11.

P R O P.

PROP. XXIV.

If a solid *AB* be contained under parallel planes, the opposite planes thereof (*AG*, *DB*, &c.) are like and equal parallelograms.



a 16. 11.

The plane *AC* cutting the parallel planes *AG*, *DB*, makes the sections *AH*, *DC*, parallels. and by the same reason *AD*, *HC* are parallels. Therefore *ADCH* is a pgr. By the like argument the other planes of the parallelepipedon are *b* pgrs. wherefore being *AF* is parallel to *HG*, and *AD* to *HC*, the angle *FAD* shall be = *GHC*. therefore because *AF* = *HG*, and *AD* = *HC*, and so *AF*. *AD* :: *HG*. *HC*. the triangles *FAD*, *GHC* are like and equal; and consequently the pgrs. *AE*, *HB* are like and equal. and the same may be shewn of the rest opposite planes. therefore, &c.

b 35. def. 1.
c 10. 11.
d 34. 1.
e 7. 5.
g 6. 6.
h 4. 1.
k 6. ax. 1.

PROP. XXV.

If a solid Parallelepipedon *ABCD* be cut by a plane *EF* parallel to the opposite planes *AD*, *BC*; then as the base *AH* is to the base *BH*, so shall solid *AHD* be to solid *BHC*.



Conceive the parallelepipedon to be extended on either side. and take *AI* = *AE*, and *BK* = *EB*, and put the planes *IQ*, *KP*, parallel to the planes *AD*, *BC*; then the pgrs. *IM*, *AH*. and *DL*, *DG*, and *IQ*, *AD*, *EF*, &c. are like and equal. where.

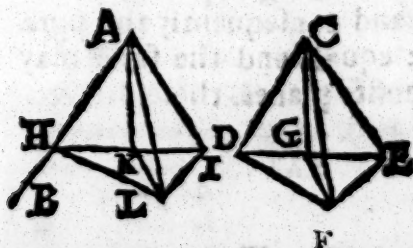
a 36. 1 and
1 def. 6.
b 24. 11.
c 10. def.
11.

- c 10. def. 11. c wherefore the Parallelepipedon AQ is = AF; and by the same reason the Parallelepipedon BP = BF. therefore the solids IF, EP are as multiplex of the solids AF, EC, as the bases IH, KH, are of the bases AH, BH. And if the basis IH be d 24. 11. $\square, =, \square$ KH, & likewise shall the solid IE be and 9 def. $\square, =, \square$ EP. & consequently AH. BH :: AF. EC. 11. Which was to be demonstrated.
- e 6. def. 5. The same may be accommodated to all sorts of prisms, whence

Coroll.

If any prisme whatsoever be cut by a plane parallel to the opposite planes, the section shall be a figure equal and like to the opposite planes.

P R O P. XXVI.



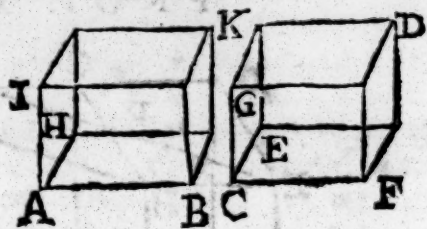
Upon a right line given AB, and at a point given in it A, to make a solid angle A-HIL equal to a solid angle given CDEF.

- From some point F a draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angle HAI = DCE, and AI = CE; and in the place HAI make the angle HAK = DCG, and AK = CG. then erect KL perpendicular to the plane HAI, and let KL be = GF. and draw AL: then AHIL shall be a solid angle equal to that given CDEF. For the construction of this does wholly resemble the framing of that, as may easily appear to any that examine it.

P R O P.

P R O P. XXVIII.

Upon a right line given AB to describe a parallelepipedon AK , like, and in like manner situate, with a solid parallelepipedon given CD .

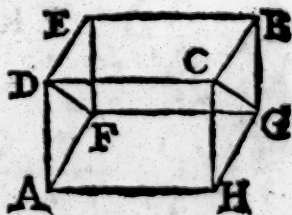


Of the plane angles, BAH , HAI , BAI , which are equal to FCE , ECG , FCG , a make the solid a 26. 11. angle A equal to the solid angle C . also b make b 12. 6. $FC.CE :: BA.AH$. b and $CE.CG :: AH.AI$ (c 22. 5. whence of equality $FC.CG :: BA.AI$) and finish the parallelepipedon AK , which shall be like to that which is given.

For by the construction, the Pgr. d BH is like d 1. def. 6. to FE , and d HI to EG , and d BI to FG , & e so e 14. 11. the opposites of these to the opposites of them: therefore the six planes of the solid AK are like to the six planes of the solid CD , f and consequently AK , CD , are like solids. f 9. def. 11. Which was to be demonstrated.

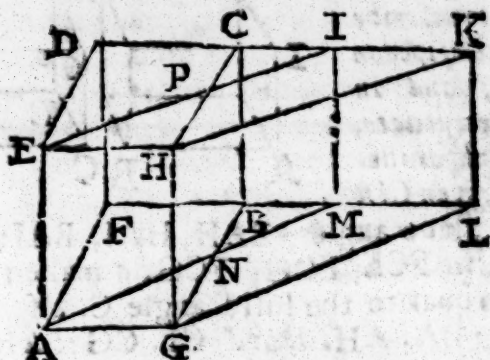
P R O P. XXVIII.

If a solid parallelepipedon AB be cut by a plane $FGCD$ drawn by the diagonal lines. DF , CG , of the opposite planes AE , HB , that solid AB shall be equally bisected by the plane $FGCD$.



For because DC , FG , are a equal and parallels, a 24. 11. b the plane $FGCD$ is a Pgr. and being a the Pgrs. b 34. 1. AE , HB , are equal and like, b also the triangles AFD , HGC , CGB , DFE are equal and like. But the Pgrs. AC , AG , are equal and like to FB and FD . therefore all the planes of the prisme $FGC-DAH$ are equal and like to all the planes of the prisme $FGCDEB$, and c consequently this prisme is equal to that. c 9. def. 11. Which was to be demonstrated.

P R O P.

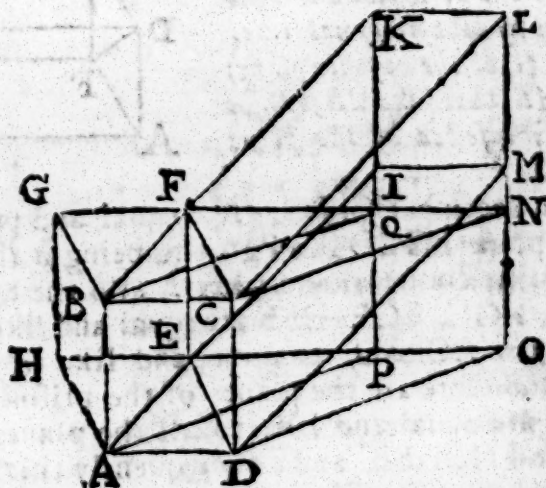


Solid parallelepipeds $AGHEFBCD$, $AGHEMLKI$, being constituted upon the same base $AGHE$, and * in the same height, whose insisting lines AF , AM , are placed in the same right lines AG , FL , are equal one to the other.

* i. e. between the parallel planes $AGHE$, $FLKD$. and so understand it in the foll. a 10. def. 11. and 35. 1. b 3. and 2. ax. 1.

For a if from the equal prisms $AFMEDI$, $GBLHCK$, the common prism $NBMPCI$ be taken away, and the solid $AGNEHR$ be added, the Parallelepiped. $AGHEFBCD$ shall be $= AGAEMLKI$. Which was to be demonstrated.

P R O P. XXX.

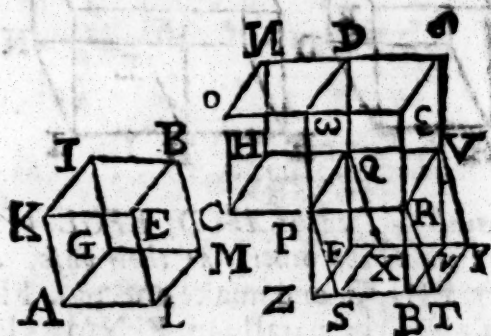


Solid parallelepipeds $ADEBCEFG$, $ADCBIMLK$ being

being constituted upon the same base $ADBC$, and in the same height, whose insisting lines AH , AI are not placed in the same right lines, are equal one to the other.

For produce the right lines HEO , GFN , and LMO , KIP ; and draw AP , DO , BQ , CN . *a* then shall DC , AB , HG , EF , PQ , ON be as well equal and parallel one to the other as AD , HE , GF , BC , KL , IM , QN , PO . *b* wherefore the parallelepipedon $ADCBPONQ$ shall be equal to either parallelepipedon $ADCBHEFG$, $ADCBIMLK$; and consequently these two are equal one to the other. Which was to be demonstrated.

PROP. XXXI.



Solid parallelepipedons, $ALEKGMBI$, $CP.OH.Q.DN$, being constituted upon equal bases $ALEK$, $CP.O$, and in the same height are equal, one to the other.

First, let the parallelepipedons AB , CD , have the sides perpendicular to the bases, and at the side CP being produced, *a* make the pgr. $PRTS$ equal and like to the pgr. $KELA$. *b* and to the parallelepipedon $PRTSQVYX$ equal and like to the parallelepipedon AB . Produce $O.E$, ND , PZ , DQF , ERB , PV , TSZ , YXF ; and draw EA , BY , ZF .

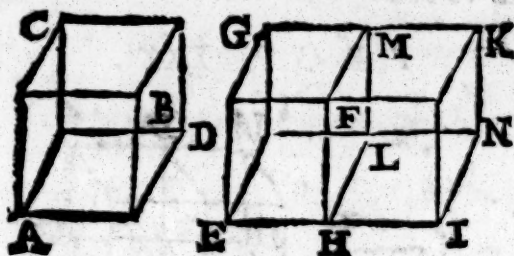
The planes $CA.N$, $CRVH$, $ZTYF$, *c* are parallels one to the other; *d* and the pgrs. $ALEK$, $CD.O$, $PRTS$, $PRBZ$ are equal. Therefore since the parallelepipedon $CD.PV$ *e* :: CA ($PRBZ$) P *e* :: e 25. 11.

*by height
understand
the perpen-
dicular
drawn from
the plane
of the base
to the op-
posite plane
a 18. 6.
b 27. 11.
and 10. def.
11.

c 30. def.
11.
d hyp. and
35. 1.
e 25. 11.

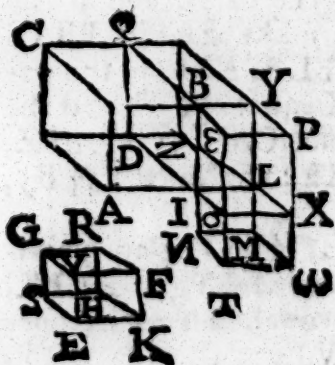
- e 25. 11. $e ::$ parallelepipedon $PRBZQV\gamma F.PV\delta\omega$; the pa-
 f 9. 5. rallelepipedon $CD f$ shall be $= PRBZQV\gamma F g$
 g 29. 11. $= PRVQSTYX h = AB$. Which was to be dem.
 h constr. But if the parallelepipedons AB, CD , have sides
 oblique to the base, then on the same bases and
 in the same height place parallelepipedons whose
 k 29. 11. sides are perpendicular to the base. k They shall
 be equal to one another, and those that are ob-
 m 1. 2x. 1. lique, m whence also the oblique parallelepipedons
 AB, CD are equal. Which was to be dem.

P R O P. XXXII.



- Solid parallelepipedons $ABCD, EFGH$, of the same
 height, are one to the other, as their base, AB, EF .
 a 45. 1. Produce EHI , a and make the pgr. $FI = AB$,
 b 31. 1. and b compleat the parallepp. $FINM$. It is clear
 c 31. 11. that the parallepp. $FINM$. ($c ABCD$) $EFGH d ::$
 d 25. 11. $FI (AB.) EF$. Which was to be demonstrated.

P R O P. XXXIII.



- a 3. 1.
 b 27. 11.

Like solid parallelepipeds,
 $ABCD, EFGH$, are
 in tripled proportion one to
 the other, of that in which
 their homologous sides or of
 like proportion AI, EK , are.

Produce the right lines
 AIL, DIO, BIN , and a
 make IL, IO, IN , equal
 to EK, KH, KF , b and
 10

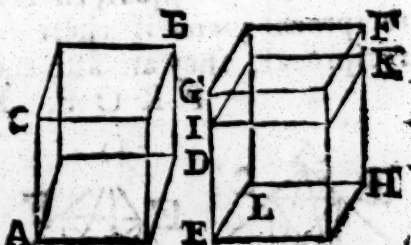
to the parallepp. IXMT equal and like to the parallepp. EFGH. *c* Let the parallepps. IXPB, DL- *c* 31. 1.
YQ be finished. *d* Then shall be AL. IL (EK) :: *d* hyp.
DI. IO (HK) :: BI. IN. KF. *e* that is the pgr. AD. *e* 1. 6.
DL :: DL. IX :: BO. IT. *f* i. e. the parallepp. *f* 32. 11.
ABCD. DLQY :: DLQY. IXPB :: IXPB. IXMT.
(*g* EFGH.) *h* therefore the proportion of ABCD *g* constr.
to EFGH is triple of the proportion of ABCD to *h* 10. def. 5.
DLQY, *k* or of AI to EK. Which was to be dem. *k* 1. 6.

Coroll.

Hence it appears that if four right lines be continually proportional, as the first is to the fourth, so is a parallelepipedon described on the first to a parallelepipedon described on the second, being like and in like manner described.

P R O P. XXXIV.

In equal solid parallelepipedons ADCB, EHGF, the bases and altitudes are reciprocal (AD. EH :: EG. AC.) And solid parallelepipedons ADCB, EHGF, whose bases and altitudes are reciprocal, are equal.



First, let the sides CB, GE be perpendicular to the bases; then if the altitudes of the solids are equal, the bases also shall be equal, and the thing is clear. But if the altitudes are unequal, from the greater EG take EI = AC, and at I draw the plane IK parallel to the base EH. then

1. Hyp. AD. EH *c* :: parallepp. ADCB. EHIK *d* 3. 1.
d :: parallepp. EHGF. EHIK *c* :: GL. IL *e* :: GE. *b* 31. 1.
IE. (f AC.) *g* it is plain therefore that AD. EH :: *c* 32. 11.
GE. AC. Which was to be demonstrated. *d* 17. 5.
e 1. 6.
2. Hyp. ADCB. EHIK *h* :: AD. EH *k* :: EG. *h* 32. 11.
EI :: GL. IL *m* :: parallepp. EHGF. EHIK. *n* *k* hyp.
wherefore the parallelepipedon ADCB = EHGF. *l* 1. 6.
Which was to be demonstrated. *m* 32. 11.
n 9. 5.

T

More.

Moreover, let the sides be oblique to the bases, and erect right parallelepipeds upon the same bases in the same altitude; the oblique parallelepipeds shall be equal to them. Wherefore since by the first part, the bases and altitudes of those be reciprocal, the base and altitudes of these also shall be reciprocal. Which was to be demonstrated.

Córoll.

All that hath been demonstrated of parallelepipeds in the 29, 30, 31, 32, 33, 34. Prop. does also agree to triangular prisms, which are half parallelepipeds, as appears by Prop. 28. Therefore,

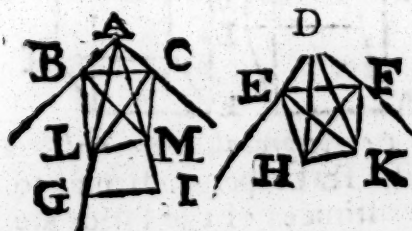
1. Triangular prisms are of equal height with their bases.

2. If they have the same or equal bases and the same altitude, they are equal.

3. If they be like, their proportion is treble to that of their sides of like proportion.

4. If they be equal, their bases and altitudes are reciprocal; and if their bases and altitudes be reciprocal, they are also equal.

P R O P. XXXV.



If there be two plane angles BAC , EDF , equal, and from the point of those angles two right lines AG , DH , be elevated on high, containing equal angles

with the lines first given, each to his correspondent angle (the angle $GAB = HDE$, and $GAC = HDF$.) and if in those elevated lines AG , DH , some points be taken, G , H ; and from these points perpendicular lines GI , HK , drawn to the planes BAC , EDF , in which the angles first given are, and right lines AI , DK , be drawn to the angles first given from the points I , K , which are made by the perpendiculars in the planes; those right lines with the elevated lines AG , DH shall contain equal angles GAM , HDK .

Make

Make DH, AL, equal; and GI, LM parallels.
 and MC to AC, MB to AB, KF to DF, KE to
 DE perpendicular; and draw the right lines BC,
 LB, LC, and EF, HF, HE; *a* and LM is perpendi- a 8. 11.
 cular to the plane BAC; *b* wherefore the angles b 3. def.
 LMC, LMA, LMB; and by the same reason the 11.
 angles HKF, HKD, HKE are right angles. There-
 fore ALq *c* = LMq + AMq *c* = LMq + CMq + ACq c 47. 1.
c = LCq + ACq. *d* therefore the angle ACL is a d 48. 1.
 right angle. Again ALq *e* = LMq + MAq *e* = LMq e 47. 1.
 + BMq + BAq *e* = BLq + BAq. *d* therefore the angle
 ABL is also a right angle. By the like inference
 the angles DFH, DEH are right angles; *f* there- f 26. 1.
 fore AB = DE, *f* and BL = EH, *f* and AC = DF,
 and CL = FH. *g* wherefore also BC = EF; *g* and g 4. 1.
 the angle ABC = DEF, *g* and the angle ACB =
 DFE. *b* whence the other right angles CBM, h 1. ax. 1.
 BCM, are equal to the other FEK, EFK. *k* there- k 26. 1.
 fore CM = FK, *l* and so also AM = DK. therefore l 47. 1.
 if from LAq *m* = HDq be taken away AMq = m constr.
 DKq, *n* there remains LMq = HKq. wherefore the n 47. 1. &
 triangles LAM, HDK are equilateral one to the 3. ax.
 other; *o* therefore the angle LAM = HDK. o 8. 1.
Which was to be demonstrated.

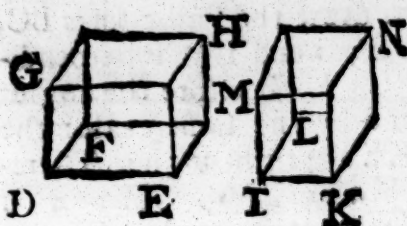
Coroll.

Therefore, if there be two plane angles equal,
 from whose points equal right lines be elevated on
 high containing equal angles with the lines first
 given, each to each; perpendiculars drawn from
 the extreme points of those elevated lines to the
 planes of the angles first given, are equal one to
 the other, viz. LM = HK.

T 2

PROP.

P R O P. XXXVI.



If there be three right lines DE, DG, DF proportional, the solid parallelepipedon DH made of them, is equal to the solid parallelepipedon IN made of the middle

line DG (IL) which is also equilateral, and equiangular to the said parallelepipedon DH .

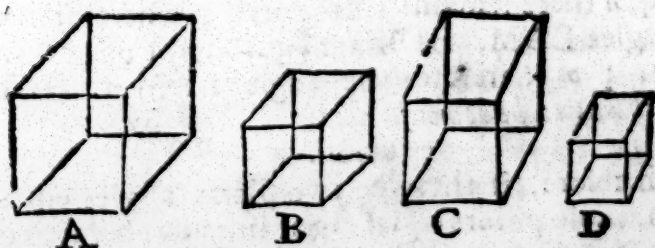
a hyp.

b 14. 6.

c 31. 11.

Because $DE.IK :: IL.DF$. *b* the pgr. LK shall be $= FE$. and by reason of the equality of the plane angles at E and I , and of the lines GD, IM , also the altitudes of the parallelepipeds are equal by the preceding Coroll. *c* therefore the parallelepps. are equal one to the other. Which was to be dem.

P R O P. XXXVII.



If there be four right lines A, B, C, D , proportional, the solid parallelepipeds A, B, C, D being like, and in like sort described from them, shall be proportional. And if the solid parallelepipeds, being like and in like sort described, be proportional ($A. B :: C. D$) then those right lines A, B, C, D , shall be proportional.

a 23. 11.

b sch. 23. 5.

For the proportions of the parallelepipeds *a* are triple of those lines; therefore if $A. B :: C. D$. *b* then shall the parallelepipedon A . parallelepipedon $B ::$ parallelepipedon C . parallelepipedon D . and so also contrarily.

P R O P.

P R O P. XXXVIII.

If a plane AB be perpendicular to a plane AC , and a perpendicular line EF be drawn from a point E in one of the planes (AB) to the other plane AC , that perpendicular EF shall fall upon the common section of the planes AD .

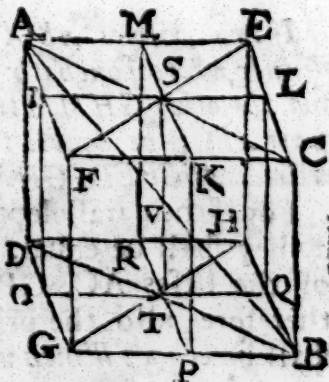


If it be possible, let F fall without the intersection AD , and in the plane AC draw FG perpendicular to AD , and join EG . The angle FGE is a right angle, and EFG is supposed to be such also; therefore two right angles are in the triangle EFG . *Which is absurd.*

a 12. 1.
b 4. and 3.
def. 11.
c 17. 1.

P R O P. XXXIX.

If the sides (AE, FC, AF, EC , and DH, GB, DG, HB) of the opposite planes AC, DB , of a solid parallelepipedon AB , be divided into two equal parts, and planes $ILQO, PKMR$, be drawn through their sections, the common section of the planes ST , and the diameter of the solid parallelepipedon AB shall divide one the other into two equal parts.



Draw the right lines SA, SC, TD, TB . Because a 34. 1. the sides DO, OT are equal to the sides BQ, QT , b 29. 1. and the alternate angles TOD, TQB equal c 4. 1. also, c the bases DT, TB , and the angles DTO, BTQ are equal, d therefore DTB is a right line. e 34. 1. and so in like manner is ASC . Moreover e as f 9. 11 and well AD is parallel and equal to FG e as FG to 1. ax. CB , and f thence AD is parallel and equal to CB ; g 33. 1.

T 3

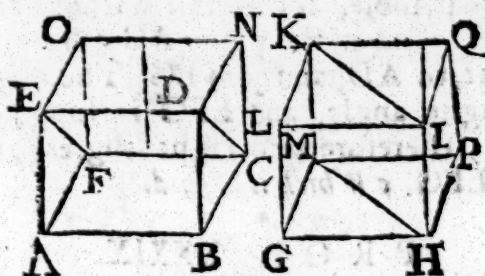
g and h 7. 11.

- g 33. r. g and consequently AC to DB *h* wherefore AB and
 h 7. r. ST are in the same plane ABCD. Therefore
 since the verticall angles AVS, BVT, and the al-
 k 7. ax. i. ternate angles ASV, BTV are equal ; *k* and
 l 26. i. AS = BT ; therefore shall AV be = BV, *l* and
 SV = VT. Which was to be demonstrated.

Coroll.

Hence in every parallelepipedon all the diame-
 ters bisect one another in one point, V.

P R O P. XL.



If two prismes *ABCFED*, *GHMLIK*, be of equal
 altitude, whereof one hath its base *ABCF* a parallelogram,
 and the other *GHM* a triangle ; and if the parallelo-
 gram *ABCF* be double to the triangle *GHM* ; those
 prismes *ABCFED*, *GHMLIK* are equal.

- a 31. r. a they shall be equal, because of the equality
 b 34. r. and of the bases AC, GP, and c of the altitudes.
 7. ax. therefore also the prismes, e the halves thereof
 c hyp. shall be equal. Which was to be demonstrated.

d 28. r.

e 7. ax. i.

Schol.

From the preceding demonstrations, the dimension
 of triangular prismes, and quadrangular, or parallele-
 pipedons, is learnt ; viz. by multiplying the altitude
 And. Tacq into the base.

As if the altitude be 10 foot, and the base 100
 square foot (the base may be measured by sch. 35.
 r. or by 41. r.) then multiply 100 by 10. and 1000
 cubic

cubic foot shall be produced for the solidity of the prisme given.

For as a rectangle, so also is a right parallelepipedon produced of the altitude multiplied into the base. Therefore every parallelepipedon is produced of the altitude multiplied into the base, as appears by 31. of this Book.

Moreover, since the whole parallelepipedon is produced of the altitude drawn into the base, the half thereof (that is, a triangular prisme) shall be produced of the altitude drawn into half the base, namely the triangle.

An Advertisement.

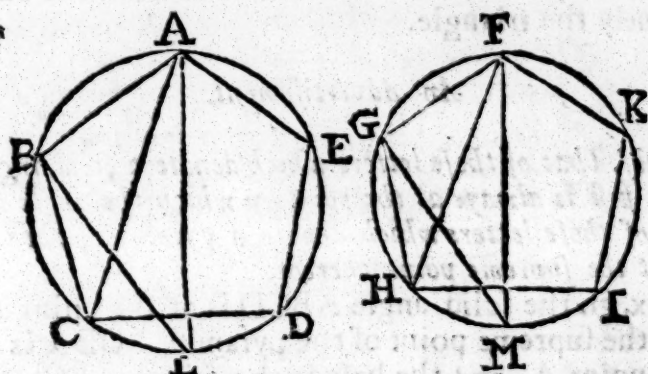
Obs. That of those letters which denote a solid angle, the first is always at the point in which the angle is; but of those letters which denote a pyramide, the last is at the supreme point thereof.

Ex. gr. the solid angle ABCD is at the point A; and the supreme point of the pyramide BCDA is at the point A. and the base is the triangle BCD.

The End of the eleventh Book.

The TWELFTH BOOK
OF
EUCLIDE'S
ELEMENTS.

PROPOSITION I.



Like polygonous figures *ABCDE*, *FGHIK* described in circles *ABD*, *FGI*, are one to another, as the squares described of the diameters of the circles *AL*, *FM*.

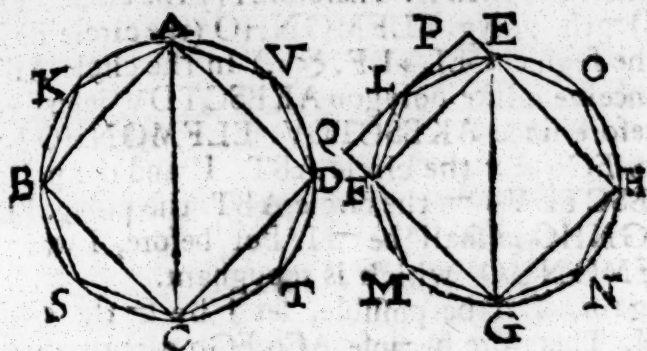
- a* 1. def. 6. Draw *AC*, *BL*, *FH*, *GM*. Because *a* the angle
b 6. 6. $\angle ABC = \angle FGH$, *a* and $AB : BC :: FG : GH$. *b* there-
c 21. 3. fore shall the angle $\angle ACB$ (*c* $\angle ALB$) be $= \angle FHG$
d 31. 3. (*c* $\angle FMG$.) but the angles $\angle ABL$, $\angle FGM$ *d* are right
e 32. 3. and so equal; *e* therefore the triangles $\triangle ABL$, $\triangle FGM$
f cor. 4. 6. are equiangular. *f* wherefore $AB : FG :: AL : FM$.
g 22. 6. *g* therefore $ABCDE : FGHIK :: AL^2 : FM^2$.

Coroll.

Hence (because $AB : FG :: AL : FM :: BC : GH$,
 &c.) the contents of like polygonous figures de-
 scribed in a circle are in proportion as the dia-
 meters.

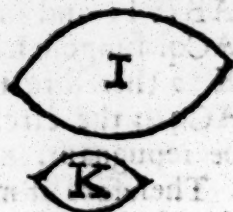
P R O P.

PROP. II.



Circles *ABT*, *EFN*, are in proportion one to another, as the squares of their diameters *AC*, *EG* are.

Suppose $AC:EG ::$ the circle *ABT*. I, I say then I is equal to the circle *EFN*.



For the first, if it be possible, let *I* be less than the circle *EFN*, and let *K* be the excess or difference. Inscribe the square *EFGH* in the circle *EFN*, *a* it being the half of a circumscribed square; and so greater than the semicircle. *b* Divide equally in two the arches *EF*, *FG*, *GH*, *HE*, and at the points of the divisions join the right lines *EL*, *LF*, &c. at *L* draw the tangent *PQ* (*c* which is parallel to *EF*) and produce *HEP*, *GFQ*. then is the triangle *ELF* *d* the half of the pgr. *EPQF*, and so greater than the half of the segment *ELF*; and in like sort the rest of those triangles exceed the halves of the rest of the segments. And if the arches *EL*, *LF*, *FM*, &c. be again bisected, and the right lines joined, the triangles will likewise exceed the half of the segments. Wherefore if the square *EFGH* be taken from the circle *EFN*, and the triangles from the other segments, and this be done continually, at length *e* there will remain some magnitude less than *K*. Let us have gone so far, namely, to the segments *EL*, *LF*, *FM*, &c. taken together

a sch. 7. 4.
b 30. 3.

c sch. 27. 3.

d 41. 1.

e r. 10.

f hyp. and together less than K. Therefore I (*f* the circle EFN
3. ax. — K) $\supset$ the polyg. ELFMGNHO (the circle EFN

— the segment EL + LF, &c.) In the circle ABT

g 30. 3. *E* g conceive a like polygon AKBSCTDV inscribed.

1. post. 1. therefore since AKBSCTDV. ELFMGNHO *h* ::

h 1. 12. ACq. EGq *k* :: the circle ABT. I. and the polyg.

k hyp. AKBSCTDV *l* $\supset$ the circle ABT. the polyg. EL.

l 9. ax. 1. FMGNHO *m* shall be $\supset$ I. but before, I was $\supset$

m 14. 5. ELFMGNHO. which is repugnant.

Again, if it be possible, let I be $\supset$ the circle

n hyp. EFN. Therefore because ACq. EGq *n* :: the circle

ABT. I; and inversely I. the circle ABT :: EGq.

ACq. suppose I. the circle ABT :: the circle EFN.

o 14. 5. K. o therefore the circle ABT $\supset$ K. p and EGq.

p 11. 5. ACq :: the circle EFN. K. which is shewn to

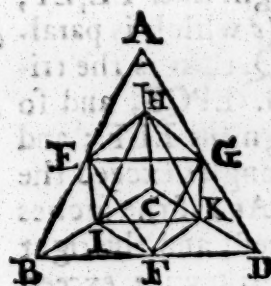
be repugnant.

Therefore it must be concluded, that I is = to
the circle EFN. Which was to be demonstrated.

Coroll.

Hence it follows, that as a circle is to a circle,
so is a polygon described in one to a like polygon
described in the other.

P R O P. III.

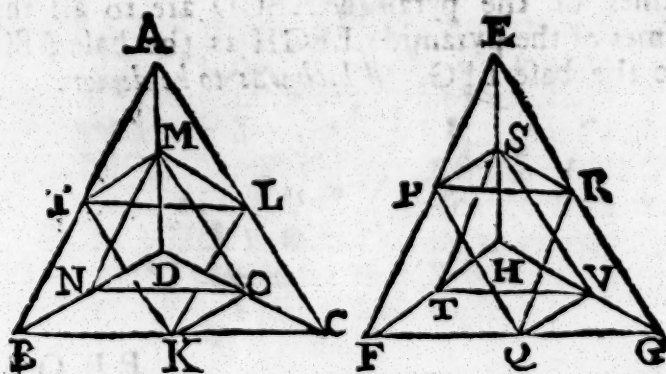


Every Pyramide ABDC having
a triangular base, may be divided
into two pyramides AEGH, HI-
KC, equal, and like one to the o-
ther, having bases triangular, and
like to the whole ABDC; and in-
to two equal prismes, BFGEIH,
FGDIHK; which two prismes
are greater than the half of the whole pyramide
ABDC.

Divide the sides of the pyramide into two parts
at the points E, F, G, H, I, K, and join the right
lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Because
the sides of the pyramide are proportionally cut,
a 2. 6. a thence HI, AB; and GF, AB; and IF, DC;
and

and HG, DC, &c. are parallels. and consequently HI, FG; and GH, FI are also parallels. therefore it is apparent that the triangles ABD, AEG, EBF, FDG, HIK, *b* are equiangular. and that the four *b* 29. 1. last are *c* equal: in like manner the triangles ACB, *c* 26. 1. AHE, EIB, HIC, FGK are equiangular; and the four last are equal one to the other. Also the triangles BFI, FDK, IKC, EGH; and lastly, the triangles AHG, GDK, HKC, EFI are like and equal. Moreover the triangles, HIK to ADB, and EGH to BDC and EFI to ADC, and FGK to ABC, *d* are paral- d 15. 11. lel. From whence it evidently follows, first, that the pyramids AEGH, HIKC are equal, and *e* like *e* 10. def. to the whole ABDC, and to one another. Next, that the solids BFGEIH, FGDIHK are prisms, and that of equal height, as being placed between the parallel planes ABD, HIK. but the base BFGE is *f* double of the base FDG. wherefore the said *f* 2. ax. 1. prisms are equal; whereof the one BFGEIH is *g* 40. 11. greater than the pyramide BEFI, that is, than AEGH, the whole than its part; and consequently the two prisms are greater than the two pyramids and so exceed the half of the whole pyramide ABDC. Which was to be demonstrated.

P R O P. IV.

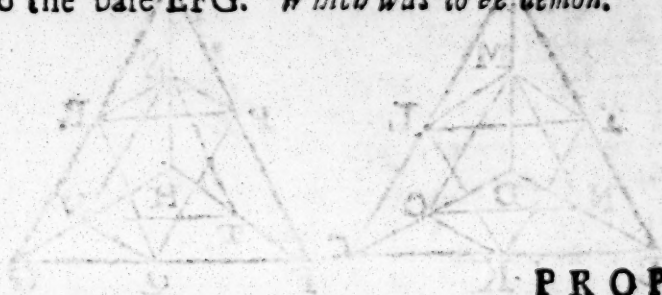


If there be two pyramids *ABCD*, *EFGH*, of the same altitude, having triangular bases *ABC*, *EFG*; and

and either of them be divided into two pyramides AL , LM , $MNOD$; and $EPRS$, $STVH$) equal one to the other and like to the whole, and into two equal prismes ($IBKLMN$, $KLCNMO$; and $PFQ RST$, $Q RGTSV$;) and if in like manner either of those pyramides made by the former division be divided, and this be done continually; then as the base of one pyramide is to the base of the other pyramide, so are all the prismes which are in one pyramide, to all the prismes which are in the other pyramide, being equal in multitude.

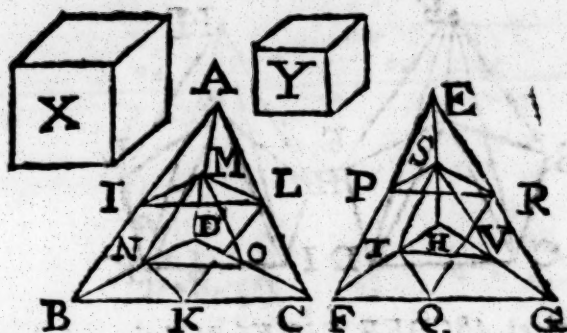
For (applying the construction of the precedent prop.) BC KC $a :: FG$. QG . b therefore the triangle ABC is to the like triangle LKC as EFG is to the like RQG . therefore by permutation ABC . EFG $d :: LKC$. RQG $e ::$ the prisme $KLCNMO$. $QRGTSV$ (for these are of equal altitude) $f :: IBKLMN$. $PFQRST$. g wherefore the triangle ABC . $EFG ::$ the prisme $KLCMNO + IBKLMN$. the prisme $QRGTSV + PFQRST$. Which was to be demonstrated.

But if the pyramides $MNOD$, $AILM$; and $EPRS$, $STVH$ be further divided; in like manner the four new prismes made hereby shall be to four produced before, as the bases MNO and AIL are to the bases STV , and EPR , that is, as LKC to RQG , or as ABC to EFG . b wherefore all the prismes of the pyramide $ABCD$ are to all the prismes of the pyramide $EFGH$ as the base ABC is to the base EFG . Which was to be demon.



PROP.

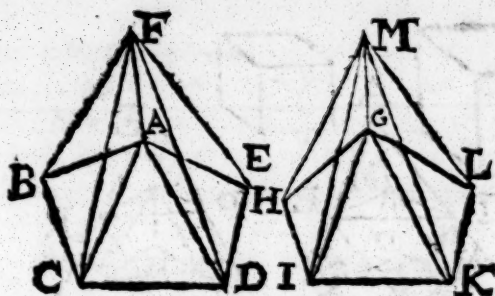
PROP. V.



Pyramides $ABCD$, $EFGH$, being of the same altitude, having triangular bases ABC , EFG , are one to another as their bases ABC , EFG , are.

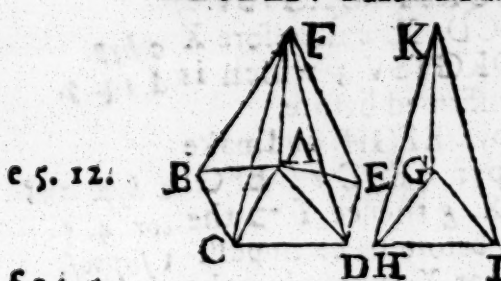
Let the triangle ABC . $EFG :: ABCD$. X . I say X is equal to the pyramide $EFGH$. For if it be possible, let X be $\supset EFGH$. and let the excess be Y . Divide the pyramide $EFGH$ into prisms and pyramides, and the other pyramides in like manner, ^a till the pyramides left $EPRS$, $STVH$, be less than the solid Y . Therefore since the pyramide $EFGH = X + Y$, it is manifest that the remaining prisms $PFQRST$, $QRGTSV$ are greater than the solid X . Conceive the pyramide $ABCD$ divided after the same manner; ^b then will be the prisme $IBKLMN + KLCNMO$. $PFQRST + QRGTSV :: ABC$. EFG ^c $::$ the pyr. $ABCD$. X . ^d therefore X ^{c hyp.} $\supset$ the prisme $PFQRST + QRGTSV$; which is ^{d 14. 8.} contrary to that which was affirmed before.

Again, conceive $X \supset$ the pyr. $EFGH$. and make the pyr. $EFGH$. $Y :: X$. the pyr. $ABCD$ ^e $:: EFG$. ^{e hyp. and} ABC . Because $EFGH$ ^f $\supset X$, ^g thence $Y \supset$ the ^{cor. 4. 5.} pyr. $ABCD$. which is shewn before to be impossible. ^{f suppos.} Therefore I conclude, that X is equal to the ^{g 14. 5.} pyr. $EFGH$. Which was to be demonstrated.



Pyramides $ABCDEF, GHIKLM$, being of the same altitude, and having polygonous bases $ABCDE, GHIKL$, are to one another as their bases $ABCDE, GHIKL$ are.

Draw the right lines AC, AD, GI, GK . then is the base ABC . ACD $a ::$ the pyr. $ABCF$. $ACDF$. b therefore by composition, $ABCD$. $ACD ::$ the pyr. AB . CDF . $ACDF$. a but also ACD . $ADE ::$ the pyr. $ACDE$. $ADEF$. c therefore of equality $ABCD$. $ADE :: ABCDF$. $ADEF$. and b thence by composition $ABCDE$. $ADE ::$ the pyr. $ABCDEF$. $ADEF$. moreover ADE . GKL $d ::$ the pyr. $ADEF$. $GKLM$; and as before, and inversely GKL . GH . $IKL ::$ the pyr. $GKLM$. $GHIKLM$. c therefore again of equality $ABCDE$. $GHIKL ::$ the pyr. $ABCDEF$. $GHIKLM$. Which was to be dem.



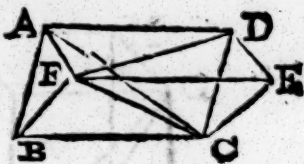
e 5. 12.

f 24. 5.

If the bases have not sides of equal multitude, the demonstration will proceed thus. The base ABC . GHI $e ::$ the pyr. $ABCF$. $GHIK$. e and ACD . $GHI ::$ the pyr. $ACDF$. $GHIK$. f therefore the base $ABCD$. $GHI ::$ the pyr. $ABCDF$. $GHIK$. e Moreover the base ADE . $GHI ::$ the pyr. $ADEF$. $GHIK$. f therefore the base $ABCDE$. $GHI ::$ the pyr. $ABCDEF$. $GHIK$.

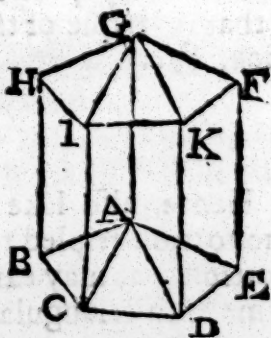
PROP. VII.

Every prisme, $ABCDEF$, having a triangular base, may be divided into three pyramids $ACBF$, $ACDF$, $CDFE$, equal one to the other, and having triangular bases.



Draw the diameters of the parallelograms, AC , CF , FD . Then the triangle ACB is $a = ACD$. a 34. 1. b therefore the pyramids of equal height $ACBF$, b 5. 12. $ACDF$ are equal. In like manner the pyr. $DFAC =$ the pyr. $DFEC$. but $ACDF$ and $DFAC$ are one and the same pyramid. c therefore the three pyramids $ACBF$, $ACDF$, $DFEC$, into which the prisme is divided, are equal one to the other. c 1. ax. 1. Which was to be demonstrated.

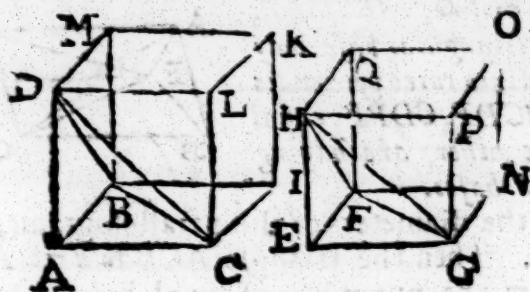
Hence, every pyramide is the third part of the prisme that has the same base and height with it, or every prisme is treble of the pyramide that has the same base and height with it.



For resolve the polygonous prisme $ABCDEFGHIK$ into triangular pyramids; and the pyramide $ABCDEH$ into triangular pyramids; a then all the parts of the prisme shall be treble a 7. 12. to all the parts of the pyramide, b consequently the b 1. 5. whole prisme $ABCDEFGHIK$ is treble to the whole pyramide $ABCDEH$. Which was to be dem.

PROP.

P R O P. VIII.



Like pyramides $ABCD$, $EFGH$, which have triangular bases ABC , EFG are in triple proportion of that in which their sides of like proportion AC , EG , are.

a 27. 11. *a* Compleat the parallelepipedons $ABICDMKL$,
 b 9. def. 11. $EFNGHQOP$, which *b* are like, and *c* sextuple of
 c 28. 11. the pyramides $ABCD$, $EFGH$. *d* and therefore in
 and 7. 12. the same proportion with them one to another,
 d 15. 5. *e* that is, triple of that of the sides of like propor-
 e 33. 11. tion, &c.

Coroll.

Hence, also like polygonous pyramides have proportion tripled to that of the sides of like proportion; as may easily be proved by resolving the same into triangular pyramides.

P R O P. IX.

See the prec. Scheme.

In equal pyramides $ABCD$, $EFGH$, having triangular bases ABC , EFG , the bases and altitudes are reciprocal; And pyramides having triangular bases, whose altitudes and bases are reciprocal, are equal.

1. Hyp. The compleated parallelepipedons $ABICDMKL$, $EFNGHQOP$ are a sextuple of the equal pyramides $ABCD$, $EFGH$ (either of either)
 a 28. 11. and 7. 12. and

and so equal one to the other, therefore the altitude (H.) the altitude (D) $b :: ABIC. EFNG$ $c :: b$ 34. 11.
 $ABC. EFG.$ Which was to be demonstrated. c 15. 5.

2. Hyp. The altitude (H.) the altitude (D) $d :: d$ hyp.
 $ABC. EFG$ $e :: ABIC. EFNG.$ f therefore the e 15. 5.
 parallelepipeds $ABICDMKL, EFNGHQOP$ f 34. 11.
 are equal. g consequently also the pyramids $AB-$ g 6. ax. 1.
 $CD, EFGH$ being subsexuple of the same, are equal. Which was to be demonstrated.

The same is applicable to polygonous pyramids, for they may also in like manner be reduced to triangulars.

Coroll.

Whatsoever is demonstrated of pyramids in prop. 6, 8, 9 does likewise agree to any sort of prisms; seeing they are triple of the pyramids that have the same base and altitude with them. Therefore

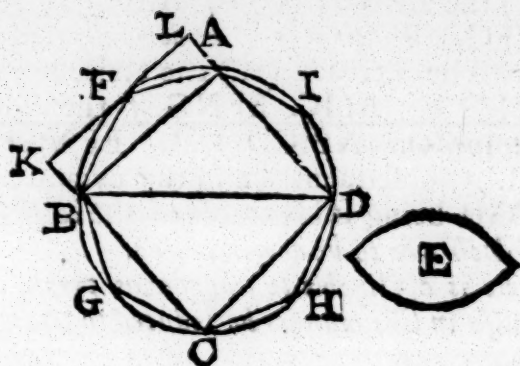
1. The proportion of prisms of equal altitude is the same with that of their bases.
2. The proportion of like prisms is triple of that of the sides of like proportion.
3. Equal prisms have their bases and altitudes reciprocal; and prisms which are so reciprocal; are equal.

Schol.

From what is hitherto demonstrated the dimension of any prism and pyramids may be collected.

a The solidity of a prism is produced of the altitude multiplied into the base; b and therefore likewise that of a pyramid, of the third part of the altitude multiplied into the base. a 1. cor. 12. b & sch. 40. 11. b 7. 12.

P R O P. X.



Every Cone is the third part of a cylinder having the same base with it $ABCD$, and the altitude equal.

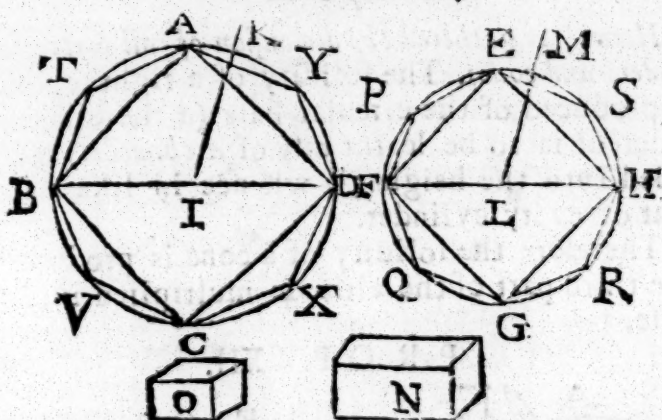
See the second figure of this Book. If you deny it, then first let such cylinder be more than triple to the cone, and let the excess be E . A prisme described on a square in the circle $ABCD$ a is subduple of a prisme described upon a square about the circle, being equal to it and the cylinder in height. Therefore a prisme upon the square $ABCD$ exceeds the half of the cylinder; and likewise a prisme upon the base AFB , of equal height to the cylinder, b is greater than the half of the segment of the cylinder AFB . continue an equal bisection of the arches, and subtract the prismes till the remaining segments of the cylinder, namely at AF , FB , &c. become less than the solid E . Therefore the cyl. — segm. AF , FB , &c. (the prisme on the base $AFBGCHDI$) c is greater than the cylinder — E (d the triple of the cone.) *c 5. ax. 1. d hyp.* therefore the pyramid, e a third part of the said prisme (being placed on the same base, and of the same height) is greater than the cone of equal height on the base $ABCD$ a circle, i. e. the part greater than the whole. *e cor. 7. 12. Which is absurd.*

But if the cone be affirmed to be greater than the third part of the cylinder, then let the excess be E . Detract the pyramids from the cone, as you did in the first part the prismes from the cylinder,

till

till some segments of the cone remain, conceive at AF, FB, BG, &c. less than the solid E. therefore the cone — E ($f \frac{1}{3}$ of the cylin.) $\supset$ the pyr. *f hyp.* AFBGCHDI (the cone — seg. AF, FB, &c.) therefore the prisme triple to the pyramid (viz. of equal height, and on the same base) is greater than the cylinder on the base ABCD, the part than the whole. *Which is abs.* Wherefore it must be granted, that the cylinder is equal to triple of the cone. *Which was to be demonstrated.*

P R O P. XI,



Cylinders and Cones *ABCDK*, *EFGHM*, being of the same altitude, are to one another as their bases *ABCD*, *EFGH* are.

Let the circle *ABCD*. the cir. *EFGH* :: the cone *ABCDK*. *N*. I say *N* is equal to the cone *EFGHM*.

For if it be possible, let *N* be $\supset$ the cone *EFGHM*, and let the excess be *O*. The preparation and argumentation of the prec. prop. being supposed; then shall *O* be greater than the segments of the cone *EP*, *PF*, *FQ*, &c. and so the solid *N* $\supset$ the pyr. *EPFQGRHSM*. In the circle *ABCD* *a* make a like polygonous fig. *ATBV CXDY*. Because the pyr. *ABVYK*. the pyr. *EFQSM* *b* :: the polyg. *ATBVY*. the polyg. *EPFQS* *c* :: the cir. *ABCD*. the cir. *EFGH* *d* :: the cone *ABCDK*. *N*. *e* thence the pyr. *EPFQGRHSM* shall be $\supset$ *N*. contrary

f hyp. and
by inver-
sion.

g 14. 5.

to what was affirmed before. Again conceive N $\sqsubset$ the cone $EFGHM$. and make the cone $EFGHM$. $O :: N$. the cone $ABCDK$ $f ::$ the circ. $EFGH$. $ABCD$. g therefore $O \sqsupset$ the cone $ABCDK$; which is absurd, as appears by what is shewn in the first part.

Therefore rather admit $ABCD$. $EFGH ::$ the cone $ABCDK$. $EFGHM$. Which was to be dem.

The same may be demonstrated of cylinders, if cylinders and prismes be conceived in the place of cones and pyramids. therefore, &c.

Schol.

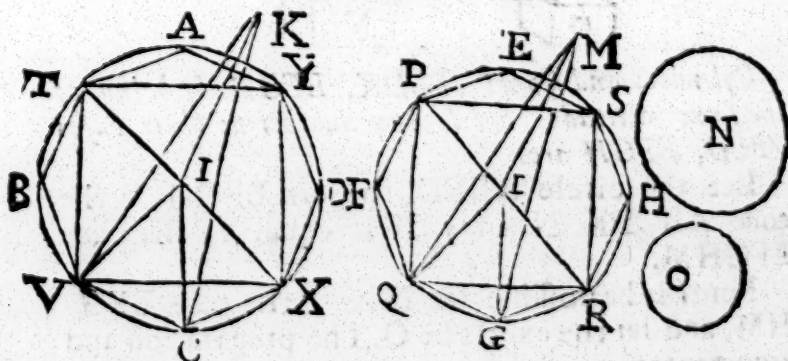
a 1. Prop.
de demenf.
cir.

b 11. 12.

Hence, is gathered the dimension of all sorts of cylinders and cones. The solidity of a right cylinder is produced of the circular base (a the dimension whereof is to be learnt out of Archimedes) multiplied into the height; b whence in like manner that of every cylinder.

Therefore the solidity of a cone is produced of the third part of the altitude multiplied into the base.

P R O P. XII.



Like cones and cylinders $ABCDK$, $EFGHM$, are in triple proportion of that of the diameters TX , PR , of their bases $ABCD$, $EFGH$.

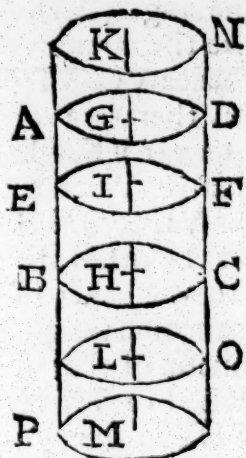
Let the cone A have to N triple proportion of TX to PR . I say N is $=$ the cone $EFGHM$. For if it be possible let N be $\sqsupset$ $EFGHM$. and let the excess be O . therefore $N \sqsupset$ the pyr $EPFQG$ - $RHSM$. Let the axes of the cones be IK , LM ,
and

and join the right lines VK, CK, VI, CI, and QM,
 GM, QL, GL. Because the cones are like, *a* a 14. def.
 thence VI. IK :: QL. LM. but the angles VIK, 11.
 QLM *b* are right angles. *c* therefore the triangles b 18. def.
 VIK, QLM are equiangular, *d* whence VC. VI :: 11.
 QG. QL. also VI. VK :: QL. QM. therefore of c 6. 6.
 equality VC. VK :: QG. QM. *e* moreover VK. CK d 4. 6.
 :: QM. MG. therefore again of equality VC. CK. e 7. 5.
 :: QG. GM. *f* therefore the triangles VKC, QMG f 5. 6.
 are like : and by the same reason the other tri-
 angles of this pyramid are like to the other of
 that. *g* wherefore the pyramids themselves are g 9. def 11.
 like. *h* But they are in triple proportion of that of h cor. 8. 12.
 VC to QG, *k* that is, of VI to RL, *l* or TX to k 4. 6.
 PR. *m* therefore the pyr. AIBVCXDYK. the pyr. l 15. 5.
 EPFQGRHSM :: the cone ABCDK. N. *n* whence m hyp. and
 the pyr. EPFQGRHSM $\supset$ N. which is repug- i . 5.
 nant to what was affirmed before. n 14. 5.

Again take N $\sqsubset$ the cone EFGHM. make the
 cone EFGHM. O :: N the cone ABCDK *o* :: the o before.
 pyr. EPRM. ATCK *p* :: GQ. VC thrice :: *p* PR. and inver-
 TX thrice. but O $\supset$ is $\supset$ ABCDK. which was sely.
 before shewn to be repugnant. Wherefore N = p cor. 8. 12.
 the cone EFGHM. Which was to be dem. q 4. 6.

But forasmuch as what proportion soever cones r 14. 5.
 have, also cylinders, being triple of them, have
 the same ; therefore cylinder to cylinder shall have
 proportion triple of the diameters of the bases.

P R O P. XIII.

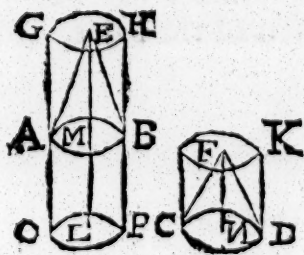


If a cylinder $ABCD$ be divided by a plane EF parallel to the opposite planes BC, AD , then as one cylinder $AEFD$ is to the other cylinder $EBCF$, so is the axis GI to the axis IH .

The axis being produced, take $GK = GI$, and $HL = IH = LM$. and conceive planes drawn at the points K, L, M , parallel to the circles AD, BC . therefore the cylinder $FD =$ the cyl. AN . and the cyl. $EC =$

the cyl. EN . and the cyl. $EC =$ the cyl. AN . and the cyl. $EC =$ the cyl. EN . therefore the cylinder EN is as multiplex of the cylinder ED as the axis IK is of the axis IG . and in like manner the cylinder EP is as multiplex of the cylinder BF , as the axis IM is of the axis IH . but as IK is $=, <, >$ IM , so is the cylinder $EN =, <, >$ EP . therefore the cylinder $AEFD$. the cyl. $EBCF :: GI. IH$. Which was to be demonstrated.

P R O P. XIV.



Cones AEB, CFD , and cylinders AH, CK , consisting upon equal bases AB, CD , are to one another as their altitudes ME, NF are.

The cylinder HA , and the axis EM being produced, take $ML = FN$; and at

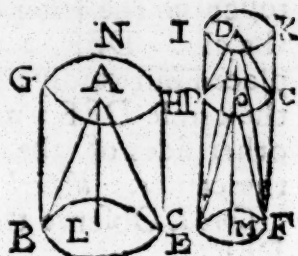
the point L draw a plane parallel to the base AB .

a 11. 12. a then shall the cyl. AP be $= CK$. b but the cyl. b 13. 12. $AH. AP(CK) :: ME. ML(NF)$. Which was to be dem. * apply 9, The same may be affirmed of cones subtriple of and 7. 12. cylinders; * as also of prisms and pyramids.

P R O P.

PROP. XV.

In equal cones BAC , EDF , and cylinders BH , EK , the bases and altitudes are reciprocal ($BC \cdot EF :: MD \cdot LA$) And cones and cylinders, whose bases and altitudes are reciprocal, are equal one to the other.



If the altitudes be equal then the bases are equal too, and the thing is evident. If unequal, then take away $MO = LA$.

1. Hyp. Then is $MD \cdot MO$ ($a LA$) $b ::$ the cyl. a 14. 12. $EK \cdot (cBH)$ $EQ d ::$ the cir. $BC \cdot EF$. Which was b constr. to be demonstrated. c hyp.

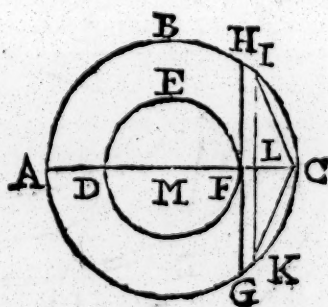
2. Hyp. $BC \cdot EF e :: DM \cdot OM$ (LA) $f ::$ the cyl. d 11. 12. $EK \cdot EQ g :: BC \cdot EF h :: BH \cdot EQ. k$ Therefore the e hyp. cylind. $EK = BH$. Which was to be dem. f 11. 12.

The same argument may be used for cones.

g 11. 5.
 h 11. 12.
 k 9. 5.

PROP. XVI.

Two unequal circles $ABCG$, DEF , having the same center M , to inscribe in the greater circle $ABCG$ a polygonous figure of equal and even sides, which shall not touch the lesser circle DEF .



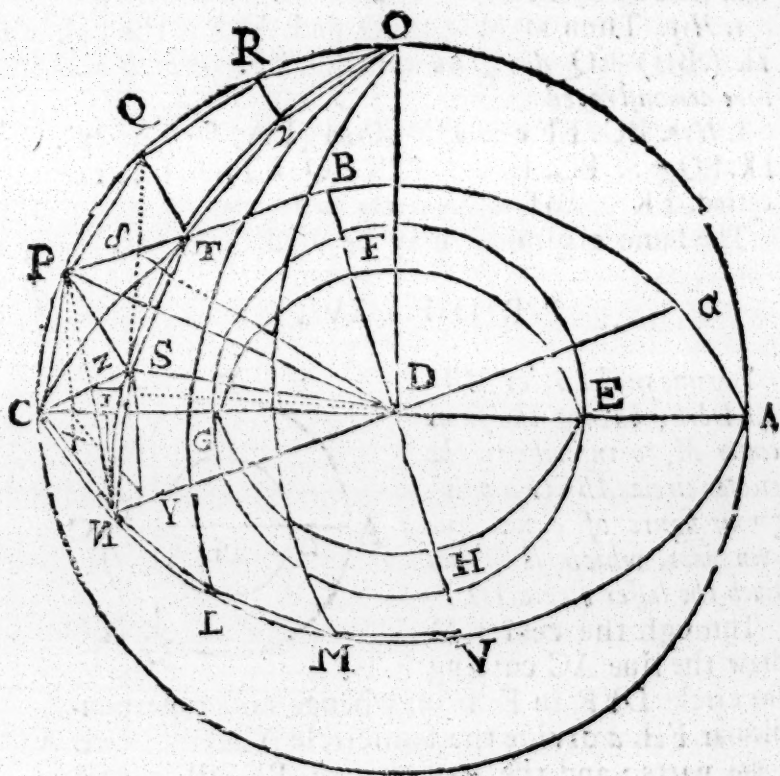
Through the center M draw the line AC cutting the circle DEF in F , from whence raise a perpendicular FH . a divide the semicircle ABC into two equal parts; and the half thereof BC also; and so do continually, b till the arch IC become less than the arch HC . from I let fall the perpendicular IL . It is manifest that the arch IC measures the whole

a 30. 3.
 b 1. 10

whole circle, and that the number of arches is even,
c sch. 16.4. and so that the subtended line IC is the side *c* of
 the polygon that may be inscribed without
d cor. 16.3. touching the lesser circle DEF. For HG *d* touches
e 28.1. the circle DEF, *e* to which IK is parallel, and
f 34. def. 1. placed outwardly; *f* wherefore IK does not touch
 the circle DEF; much less do CI, CK, and the
 other sides of the polygon more remote from
 the center. Which was to be done.

Coroll. Observe that IK touches not the circle DEF.

P R O P. XVII.



Two spheres *ABCV*, *EFGH*, consisting about the same
 center *D*, being given, to inscribe a solid of many sides
 (or Polyedron) in the greater sphere *ABCV*, which
 shall

shall not touch the superficies of the lesser sphere EFGH.

Let both the spheres be cut by a plane passing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV, *a* inscribe the equilateral polygone VMLNC, &c. not touching the circle EFGH: then draw the diameter Na, and erect DO perpendicular to the plane ABC. by DO, and by the diameters AC, Na, conceive planes DOC, DON erected, which shall be *b* perpendicular to the circle ABCV, and so in the superficies of the sphere make *c* the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST, T₂, O *d* be fitted, equal, and of equal multitude with CN, NL, &c. make the same construction in the other quadrants OL, OM, &c. and in the whole sphere. Then I say the thing required is done.

a 16. 12.

b 18. 11.

c cor. 33. 6

d 4. 1.

From the points P, S, to the plane ABCV draw the perpendiculars PX, SY, *e* which shall fall on the sections AC, Na. Therefore because both the right angles PXC, SYN, *g* and PCX, SNY insisting on *b* equal circumferences, *f* are equal, the triangles also PCX, SNY *h* are equiangular. Wherefore being PC *k* = SN, *l* also is PX = SY, *k* *l* and XC = YN; *m* whence DX = DY. *n* and therefore DX. XC :: DY. YN. *o* therefore YX, NC are parallels. but because PX, SY are equal, and since being perpendicular to the same plane ABCV, they are also *p* parallels, *q* therefore YX, SP shall be equal and parallels. *r* whence SP, NC, are parallel one to the other; and so the quadrilateral NCPS, and by the same reason SRQT, TQRG, as also the *t* triangle *γ* RO are so many planes. In like manner the whole sphere may be shewn full of such quadrilaterals and triangles. wherefore the figure inscribed is a polyedron.

e 38. 11.

f 12. ax.

g 27. 3.

h 32. 1.

k constr.

l 26. 1.

m 3. ax. 1.

n 7. 5.

o 2. 6.

p 6. 11.

q 33. 1.

r 9. 12.

s 7. 11.

t 2. 11.

From the center D *u* draw DZ perpendicular to the plane NCPS; and join ZN, ZC, ZS, ZP. Because

u 11. 11.

cause

x 4. 6. cause DN. NCx::DY.YX. thence NC is $y \sqsubset YX$
 y 14. 5. (SP,) and likewise SP $\sqsubset$ TQ, and TQ $\sqsubset$ R.
 And because the angles DZC, DZN, DZS, DZP
 z 3. def. 11. z are right, and the sides DC, DN, DS, DP, a equal,
 a 15. def. and DZ common, b thence ZC, ZN, ZS, ZP are e-
 11. qual one to the other; and consequently about the
 b 47. 1. quadrilateral NCPS c a circle may be described, in
 c 15. def 1. which (because NS, NC, CP, are d equal, and NC
 d constr. $\sqsubset$ SP) NC e subtends more than the quadrant, f
 e 28. 3. therefore the ang. NZC at the center is obtuse,
 f 33. 9. g therefore NCq $\sqsubset$ 2 ZCq (ZCq + ZNq.) Let
 g 12. 2. NI be drawn perpendicular to AC. therefore since
 h 32. 1. the angle ADN (b DNC + DCN) k is obtuse, the
 k 9. ax. 1. half of it DCN shall be greater than the half of a
 l 5. 1. right angle; and so that which remains of the right
 n 19. 1. ang. CNI shall be less than it. n whence IN $\sqsubset$ IC.
 o 47. 1. therefore NCq (NIq + ICq) o $\sqsubset$ 2 INq. there-
 p 47. 1. fore IN $\sqsubset$ ZC. and consequently QZ p $\sqsubset$ DI. but
 q cor. 19. the point I is q without the sphere EFGH. and
 12. so much more the point Z. wherefore the plane
 r 47. 1. NCPS, (whose r next point to the center is Z,) does
 not touch the sphere EFGH. And if a perpendicu-
 lar Ds be drawn to the plane SPQT, the point s,
 and so also the plane SPQT is yet further remo-
 ved from the center. which is also true of the o-
 ther planes of the polyedron. Therefore the polye-
 dron ORQPON, &c. inscribed in the greater sphere,
 does not touch the lesser. Which was to be done.

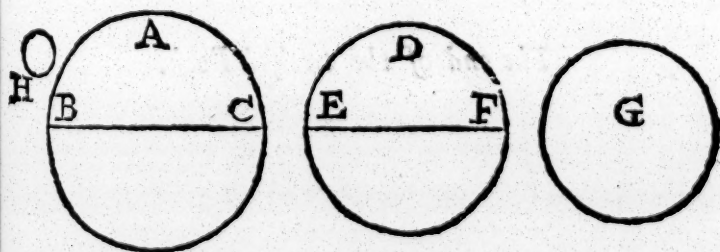
Coroll.

Hence it follows, that if in any other sphere a solid polyedron, like to the abovesaid solid polyedron, be inscribed, the proportion of the polyedron in one sphere to the polyedron in the other is triple of that of the diameters of the spheres.

For if right lines be drawn from the centers of the spheres to all the angles of the bases of the said polyedrons, then the polyedrons will be divided into pyramids equal in numbers and like; whole

whose homologous sides are semidiameters of the spheres; as appears, if the lesser of these spheres be conceived described within the greater about the same center. For the right lines drawn from the center of the sphere to the angles of the bases will agree one to the other by reason of the likeness of the bases; and so will like pyramids be made. Wherefore since every pyramid in one sphere to every pyramid like it in the other sphere *a* has proportion triple to that of the homologous sides, that is, of the semidiameters of the spheres; and *b* as one pyramid is to one pyramid, *b* 12. 5. so all the pyramids, that is, the solid polyedron composed of these, are to all the pyramids, that is, the solid polyedron composed of the others; therefore the polyedron of one sphere shall have to the polyedron of the other sphere, proportion triple of that of the semidiameters, *c* and so of *c* 15. 5. the diameters of the spheres.

P R O P. XVIII.



Spheres *BAC*, *EDF*, are in triple proportion one to the other of that in which their diameters *BC*, *EF*, are.

Let the sphere *BAC* be to the sphere *G* in triple proportion of that of the diameter *BC* to the diameter *EF*. I say $G = EDF$. For if it be possible, let *G* be $\neq EDF$. and conceive the sphere *G* concentrical with *EDF*. In the sphere *EDF* *a* 17. 12. inscribe a polyedron not touching the sphere *G*, and a like polyedron in the sphere *BAC*. These polye-

b cor. 17. polyedrons b are in triple proportion of the dia-
 12 meters BC, EF, c that is, of the sphere BAC to
 e hyp. G. d Consequently the sphere G is greater than
 d 14. 5. the polyedron inscribed in the sphere EDF,
 the part than the whole.

Again, if it be possible, let the sphere G be $\square$
 EDF. and as the sphere EDF is to another sphere
 e hyp. in- H, so let G be to BAC, e that is, in triple propor-
 vof. tion of the diameter EF to BC. therefore since
 f 14. 5. BAC $f \square$ H, we shall incur the absurdity of the
 first part. wherefore rather the sphere G = EDF.
Which was to be demonstrated.

Coroll.

Hence, as one sphere is to another sphere, so is
 a polyedron described in that to a like polyedron
 described in this.

The end of the twelfth Book.

The THIRTEENTH BOOK
OF
EUCLIDE'S
ELEMENTS.

PROPOSITION I.

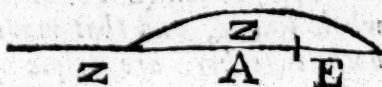
If a right line z be divided according to extreme and mean proportion ($z. a :: a. e.$) the square of the half of the whole line z , and of the greater segment a , as one line, is quintuple to that which is described of half of that whole line z .

I say $Q. a + \frac{1}{2} z =$

$5 Q. \frac{1}{2} z. a$ that is, aa

$+ \frac{1}{4} zz + za = zz +$

$\frac{1}{4} zz$ b or $aa + za = zz.$ For $ze + za c = zz.$ and d hyp. and $ze d = aa.$ e therefore $aa + za = zz.$ Which was $16. 6.$ to be demonstrated. $e 2. ax. and$
 $1. ax.$



PROP. II.

See the 1. Scheme.

If a right line $\frac{1}{2} z + a$ be in power quintuple to a segment of it self $\frac{1}{2} z.$ the line double of the said segment (z) being divided according to extreme and mean proportion, the greater segment is (a) the other part of the right line at first given $\frac{1}{2} z + a.$

I say $z. a :: a. e.$ Because by the hyp. $* aa +$ $* 4. 2.$

$\frac{1}{4} zz + za = zz + \frac{1}{4} zz;$ or $aa + za = zz$ $a =$ $a 2. 2.$

$ze + za, b$ thence shall aa be $= ze. c$ wherefore $z. a :: a. e.$ $b 3. ax. 1.$

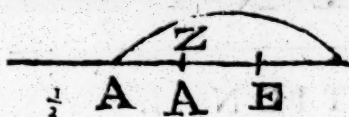
Which was to be demonstrated. $c 17. 6.$

PROP.

P R O P. III.

If a right line z be divided according to extreme and mean proportion ($z. a :: a. e.$) the line made of the less segment e and half of the greater segment a , is in power quintuple to the square, which is described of the half line of the greatest segment a .

a 4. 2.
b 3. ax.
c 3. 2.
d hyp. and
17. 6.

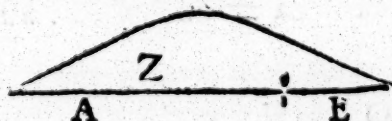


I say $Q: e + \frac{1}{2}a = 5 Q: \frac{1}{2}a. a$ that is $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$.
 b or $ee + ea = aa$. For
 $ee + ea c = zed = aa$. Which was to be dem.

P R O P. IV.

If a right line z be cut according to extreme and mean proportion ($z. a :: a. e.$) the square made of the whole line z , and that made of the lesser segment e , both together, are triple of the square made of the greater segment a .

a 4. 1.
b 3. 2.
c 17. 6.
d 2. ax.



I say $zz + ee = 3 aa. a$ or $aa + ee + 2 ae + ee = 3 aa$. For $ae + ee = ze c = aa, d$

therefore $aa + 2 ae + 2 ee = 3 aa$. Which was to be demonstrated.

P R O P. V.

D—A—C—B

If a right line AB be cut according to extreme and mean proportion in C , and a line AD , equal to the greater segment BC , added to it, the whole right line DB is divided according to extreme and mean proportion; and the greater segment is the right line AB given at the beginning.

a hyp.

For because $AB. AD :: AC. CB$. and by inversion $AD. AB :: CB. AC$. therefore by composition $DB. AB :: AB. AC (AD.)$ Which was to be dem.

Schol.

Schol.

But if $BD.BA :: BA.AD$. then shall be $BA.AD :: AD.BA - AD$. For by division is $BD - BA (AD) BA :: BA - AD.AD$. therefore inversely $BA.AD :: AD.BA - AD$.

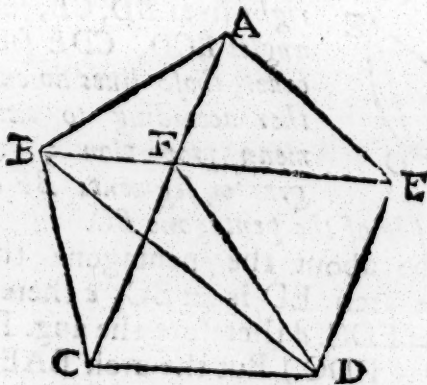
PROP. VI.

D—A—C—B If a rational right line AB be cut according to extreme and mean proportion in C , either of the segments (AC, CB) is an irrational line of that kind which is called apotome or residual.

To the greater segment AC add $AD = \frac{1}{2} AB$.
 b therefore $DCq = 5 DAq$. c therefore $DCq \sqsupset DAq$. consequently d since AB, e and so the half thereof DA are ρ , likewise DC is ρ . But because $5.1 ::$ not $Q.Q$. f thence is $DC \sqsupset DA$. g therefore $DC - AD$, that is, AC , is a residual line. Further, because $ACq h = AB \times BC$, and AB is ρ, k likewise BC is a residual line. Which was to be demonstrated.

a 3.
 b 1. 13.
 c 6. 10.
 d hyp.
 e sch. 12.
 f 10.
 g 9. 10.
 h 74. 10.
 i 17. 6.
 k 98. 10.

PROP. VII.



If three angles of an equilateral pentagon $ABCDE$, whether they follow in order, (EAB, ABC, BCD), or not, (EAB, BCD, CDE) be equal, the pentagon $ABCDE$ shall be equiangular.

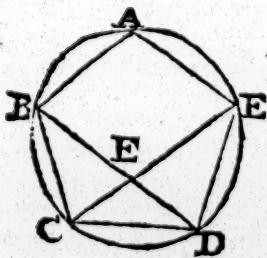
Let

Let the right lines BE, AC, BD, be subtended to the equal angles in order.

a hyp. Being the sides EA, AB, BC, CD, and the included angles *a* are equal, *b* therefore shall the bases BE, AC, BD, *c* and the angles AEB, ABE, BAC, BCA, be equal. *d* wherefore $BF = FA$, *e* and consequently $FC = FE$; therefore the triangles FCD, FED, are equilateral one to the other: *f* whence the angle $FCD = FED$. *g* consequently the ang. $AED = BCD$. In like manner the ang. CDE is equal to the rest; wherefore the pentagone is equiangular. *Which was to be dem.*

h 4. 1. But if the angles EAB, BCD, CDE, which are not in order, be supposed equal, *b* then shall the ang. AEB be $= BDC$, and $BE = BD$. *k* and thence the ang. $BED = BDE$. *l* consequently the whole ang. $AED = CDE$, therefore because the angles A, E, D, in order, are equal, as before, the pentagone shall be equiangular. *Which was to be dem.*

P R O P. VIII.



If in an equilateral and equiangular pentagone ABCDE, two right lines BD, CE, subtend two angles BCD, CDE following in order, those lines do cut one another according to extreame and mean proportion; and their greater segments BF or EF are

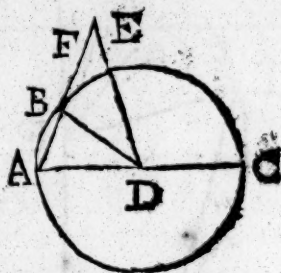
equal to the side of the pentagone BC.

a 14. 4. Describe about the pentagone the circle
b 28. 3. ABD. *b* The arch ED is $= BC$, *c* therefore the
c 27. 3. angle $FCD = FDC$. *d* therefore the ang. $BFC = 2$
d 32. 1. FCD ($FCD + FDC$.) But the arch BAE *b* is $= 2$
e 33. 6. ED, and consequently the angle BCF . *c* $= 2 FCD$
f 6. 1. $= BFC$. *f* wherefore $BF = BC$. *Which was to be dem.*
g 27. 3. Moreover because the triangles BCD, FCD, are
h 4. 6. *g* equiangular, *b* therefore $BD : DC$. (*BF*.) $:: CD$. (*BF*)

(BE.) FD. and likewise EC. $EF :: EF.FC.$
Which was to be demonstrated.

PROP. IX.

If the side of an Hexagone BE, and the side of a Decagone AB both described in the same circle ABC, be added together, the whole right line AE is cut according to extreme and mean proportion ($AE.BE :: BE.AB.$) and the greater segment therefore is the side of the Hexagone BE.



Draw the diameter ADC, and join the right lines DB, DE. Because the angle BDC $a = 4$ BDA and the angle BDC $b = 2$ DBA ($DAC + DBA$) thence shall DBA ($bDBE + BED$) c be $= 2$ BDAd $= 2$ BDE. whence the angle DBA or DAB $e = ADE$. Therefore the triangles ADE, ADB, are equiangular: f wherefore $AE.AD (g BE) :: AD.(BE.) AB.$ Which was to be demonstrated.

a hyp. and
27. 3.
b 32. 1.
c 7. ax. 1.
d 5. 1.
e 1. ax. 1.
f 4. 6.
g cor. 15. 4.

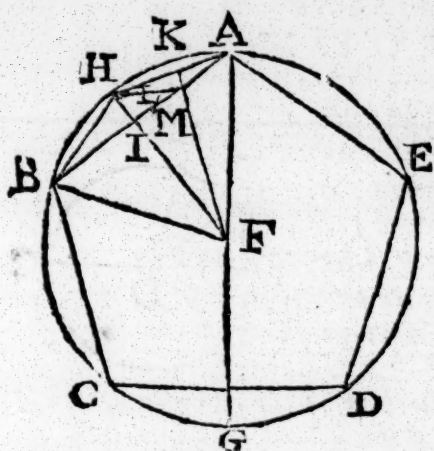
Coroll.

Hence, If the side of a hexagone in a circle be cut according to extream and mean proportion; the greater segment thereof shall be the side of the decagone in the same circle.

X

PROP.

PROP. X.



If an equilateral Pentagone $ABCDE$ be described in a circle $ABCE$, the side of the pentagone AB containeth in power both the side of a hexagone FB , and the side of a decagone AH described in the same circle.

Draw the diameter AG , and bisect equally the arch AH in K . and draw FK , FH , FB , BH , HM .
 The semicircle AG — the arch $ACa = AG$ — AD . that is, the arch $CG = GD$ $b = AH = HB$.
 therefore the arch $ECG = 2 BHK$; c and so the angle $BFG = 2 BFK$. d but the angle $BFG = 2 BAG$. e therefore the angle $BFK = BAG$.
 Wherefore the triangles BFM , FAB , f are equiangular. g whence AB , $BF :: BF$. KM . b therefore $AB \times BM = BFq$. Moreover the angle $AFK = HFK$, and $FA = FH$. m wherefore $AL = LH$, m and the angles FLA , FLH are equal, and so right angles. therefore the angle $LHM = LAM$ $n = HBA$. therefore the triangles AHB , AMH , o are equiangular; wherefore AB . $AH :: AH$. AM . q therefore $AB \times AM = AHq$.
 So that seeing $ABq r = AB \times EM + AB \times AM$,
 f thence $ABq = BFq + AHq$. Which was to be demonstrated.

Coroll.

1. Hence, a right line (FK) which being drawn from the center (F) divides an arch (HA) into two equal segments, does also divide the right

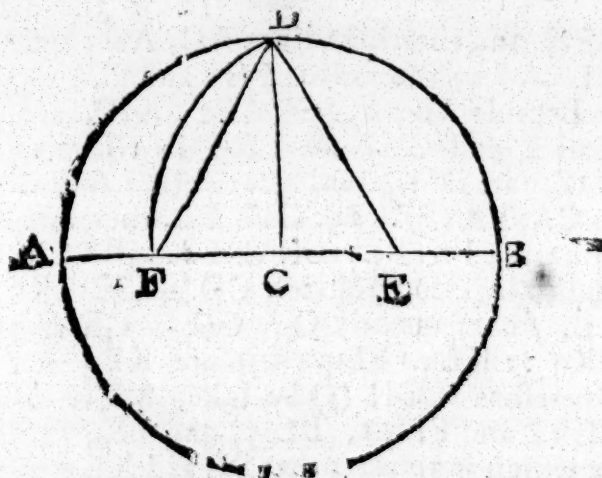
right line (HA) subtending that arch perpendicularly into two equal segments.

2. The diameter of a circle (AG.) drawn from any angle (A) of a pentagone, does divide equally in two, both the arch (CD,) which the side of the pentagone opposite to that angle subtends, and also the opposite side it self (CD) and that perpendicularly.

Schol.

Here, according to our promise, we shall lay down a ready praxis of the 11. prop. of the 4. Book.

Probl.



To find out the side of a pentagone to be inscribed in a circle ADB.

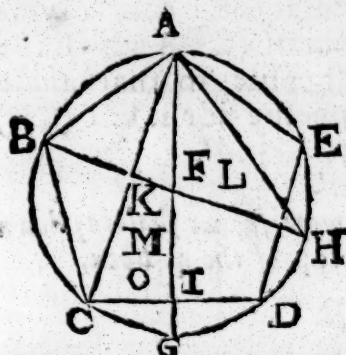
Draw the diameter AB, to which erect a perpendicular CD at the center C, divide CB equally in E, and make EF = ED. then DF shall be the side of the pentagone

For $BF \times FC + EC^2 = EF^2 = ED^2 = DC^2 + FC^2$ *a* 6. 2.
 $DC^2 + FC^2$ *d* therefore $BF \times FC = DC^2$ or *b* *constr.*
 BC^2 *e* wherefore $BF \cdot BC :: BC \cdot FC$, therefore *c* 47. 1.
 since BC is the side of a hexagone, *f* FC shall *d* 3. *ax.*
 be the side of a decagone. Consequently DF *b* *e* 17. 6.
 $= \sqrt{DC^2 + FC^2}$ *g* is the side of a pentagone. *f* 9. 13.
 Which was to be done. *g* 10. 13.

X 2

PROP. h 47. 1.

P R O P. XI.



If in a circle $ABCD$, whose diameter is rational AG , an equilateral pentagon be inscribed $ABCDE$; the side of the pentagon AB is an irrational line of that kind which is called a minor line.

Draw the diameter

* 10. 6.

BFH , and the right lines AC , AH ; and * make $FL = \frac{1}{4}$ of the radius FH ; and $CM = \frac{1}{4} CA$.

a cor. 10.

13.

b 32 1.

c 4. 6.

d 15. 5.

e 18. 5.

f 22. 6.

g 1. 13.

h 9. 10.

k 74. 10.

l 9. 10.

m cor. 8.6.

and 17. 6.

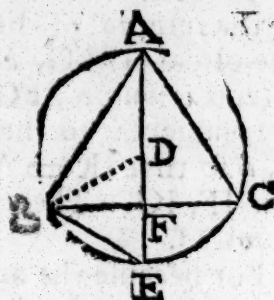
n 95. 10.

Because the angles AKF , AIC , are a right angles, and CAI common, the triangles AKF , AIC , are b equiangular: c therefore $CI. FK :: CA. FA$ (FB) d: : $CM. FL$. therefore by permutation $FK. FL :: CI. CM$ d: : $CD. CK$ (2CM) and so by e composition $CD + CK. CK :: KL. FL$. f consequently Q: $CD + CK$ (g 5 CKq) $CKq :: KLq. FLq$. therefore $KLq = 5 FLq$. wherefore if BH (ρ) be taken 8, FH shall be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears that BL and KL are $\rho b \sqsupset$, k and so BK is a residual, and KL its congruent or adjoining line. but being $BLq - KLq = 20$. l thence $BL \sqsupset \sqrt{BLq - KLq}$. m whence BK shall be a fourth residual line. Therefore because $ABq m$ is $= HB \times BK$, n shall AB be a minor line. Which was to be dem.

P R O P.

PROP. XII.

If in a circle *ABEC* an equilateral triangle *ABC* be inscribed, the side of that triangle *AB* is in power triple to the line *AD* drawn from *D* the center of the circle to the circumference.

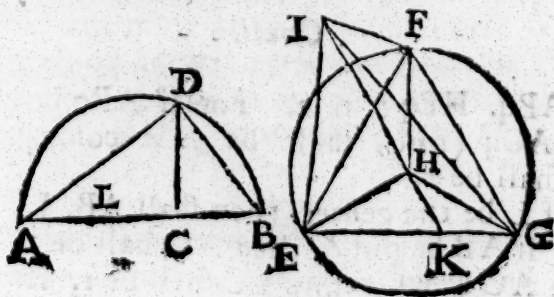


The diameter being extended to *E*, draw *BE*. Because the arch *BE* *a* = *EC*, the arch *BE* is the sixth part of the circumference, *b* therefore *BE* = *DE*, hence *A*² *c* = 4 *D*² *e* (4 *B*² *e*) *d* = *A*² *b* + *B*² *e* (+ *A*² *d*.) *e* consequently *A*² *b* = 3 *A*² *d*. Which was to be demonstrated.

Coroll.

1. *A*² *b* :: 4. 3.
2. *A*² *b* *f* :: 4. 3. *f* For *A*² *b* *f* :: *A*² *e* *f* cor. 8. 6. and 22. 6.
3. *D*² *e* = *F*² *e*. For the triangle *EBD* *g* is equilateral, *b* and *BF* perpendicular to *ED*. *h* therefore *EF* = *FD*. *g* cor. 13. 4. *h* cor. 3. 3.
4. Hence, *AF* = *DE* + *DF* = 3 *DF*.

PROP. XIII.



To describe a pyramid *EGFI*, and comprehend it in a sphere given: and to demonstrate that the diameter of the sphere *AB* is in power sesquialter of the side *EF* of the pyramid *EGFI*.

X 3

About

a 10. 6.

About AB describe the semicircle ADB; *a* and let AC be $= 2$ CB. from the point C erect the perpendicular line CD, and join AD, DB, then at the interval of the radius HE $=$ CD describe the circle HEFG, *b* wherein inscribe the equilateral triangle EFG. from H *e* erect IH $=$ CA perpendicular to the plane EFG, produce IH to K, *d* so that IK $=$ AB; and join the right lines IE, IF, IG. Then EFGI shall be the pyramid required.

b cor. 15. 4.

c 2. 11.

d 3. 1.

c constr.

f 41. 1.

g 20. 6.

h 2. ax.

k 12. 13.

l 1 ax. 1.

m 8. ax.

n 15. def. 1.

* 31. def.

11.

o cor 8. 6.

p constr.

For because the angles ACD, IHE, IHF, IHG, *e* are right angles; and CD, HE, HF, HG *e* equal, *e* and IH $=$ AC; *f* therefore AD, IE, IF, IG shall be equal among themselves. But being AC (2 CB.) CB *g* :: ACq. CDq. thence shall ACq be $= 2$ CDq. therefore ADq *f* $=$ ACq + CDq *b* $= 3$ CDq $= 3$ HEq *k* $=$ EFq. *l* therefore AD, EF, IE, IF, IG are equal, and so the pyramid EFGI is equilateral. But if the point C be placed upon H, and AC upon HI, the right lines AB, IH, *m* shall agree, as being equal. Wherefore the semicircle ADB being drawn about the axis AB or IK *n* shall pass by the points E, F, G, * and so the pyramid EFGI shall be inscribed in a sphere. Which was to be done.

Also it is manifest that BAq. ADq *o* :: BA. AC *p* :: 3. 2. Which was to be dem.

Coroll.

1. ABq. HEq :: 9. 2. For if ABq be put 9, then ACq (EFq) shall be 9. *q* consequently HEq shall be 2.

2. If L be the center, then shall AB. LC :: 6. 1. For if AB be put 6, then AL shall be 3. *r* and thence AC 4. wherefore LC shall be 1. Hence

p constr.

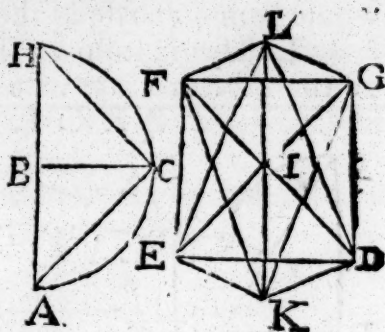
3. AB Hl :: 6. 4 :: 3. 2. whence

4. ABq. Hlq :: 9. 4.

P R O P.

PROP. XIV.

To describe an Octaedron $KEFGDL$, and comprehend it in the given sphere, wherein a pyramide is: and to demonstrate that AH the diameter of the sphere is in power double of AC the side of that Octaedron.



About AH describe the semicircle ACH . and from the center B erect the perpendicular BC . draw AC , HC ; then upon $ED = AC$ ^a make the square $EFGD$, whose diameters DF , EG , cut in the center I . from I draw $IL = AB$ ^b perpendicular to the place $EFGD$. produce IL ^c till $IK = IL$. and join KE , KF , KG , KD , LE , LF , LG , LD ; then shall $KEFGDL$ be the Octaedron required.

For AB , BH , FI , IE , &c. being semidiameters of equal squares are equal one to the other. ^d whence the bases LF , LE , FE , &c. of the right angled triangles LIE , LIF , FIE , &c. are equal, and consequently the eight triangles LFE , LEG , LGD , LDE , KEF , KFG , KGD , KDE , are equilateral, ^e and make an Octaedron, which may be inscribed in a sphere, whose center is I . and IL or AB the radius. (because AB , IL , IF , IK , &c. ^f are equal.) Which was to be done. Moreover, it is evident that AHq (LKq) $g = 2 ACq$ ($2 LDq$.) Which was to be dem.

Corol.

1. Hence it is manifest, that in the octaedron the three diameters EG , FD , LK do cut one the other perpendicularly in the center of the sphere.

2. Also that the three planes $EFGD$, $LEKG$, $LFKD$ are squares, cutting one another perpendicularly.

X 4

3. The

a 46. 1.

b 12. 11.

c 3. 1.

d 4. 1.

e 27. def. 11.

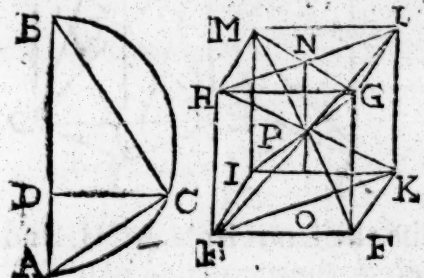
f constr.

g 47. 1.

3. The Octaedron is divided into two like and equal pyramids EFGDL, and EFGDK, whose common base is the square EFGD.

4. Lastly it follows that the opposite bases of the octaedron are parallel one to the other.

P R O P. XV.



To describe a cube EFGHIKLM, and comprehend it in the same sphere wherein the former figures were; and to demonstrate that AB the diameter of the

sphere is in power triple to EF the side of that cube.

Upon AB describe a semicircle ACB; and make $AB = 3 DA$ from D raise the perpendicular DC, and join BC and AC. Then upon $EE = AC$ make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equal to EF, and connect them with the right lines IK, KL, LM, IM. The solid EFGHIKLM, is a cube, as is sufficiently apparent from the construction.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG, by which let the planes EKLH, FIMG be drawn, cutting one another in the line NO. which shall divide equally in two parts the diameters of the cube EL, FM, GI, HK, in P the center of the cube. therefore P shall be the center of a sphere passing by the angular points of the cube. Moreover, $ELq = EKq + KLq = 3 KLq$, or $3 ACq$. but $ABq : ACq :: BA : DA$. therefore $AB = EL$. wherefore we have made a cube, &c. Which was to be done.

Coroll

1. Hence it is manifest, that all the diameters of the cube are equal one to another, and do equally bisect one another in the center of the sphere.

§ 5. 11.

2 10. 6.

b 45. 11.

c cor. 39. 11.

d 15. def.

11. and 14.

def. 11.

e 47. 1.

f constr.

g cor. 8.6.

h 14. 5.

sphere. And by the same means the right lines which conjoin the centers of the opposite squares are equally bisected in the same center.

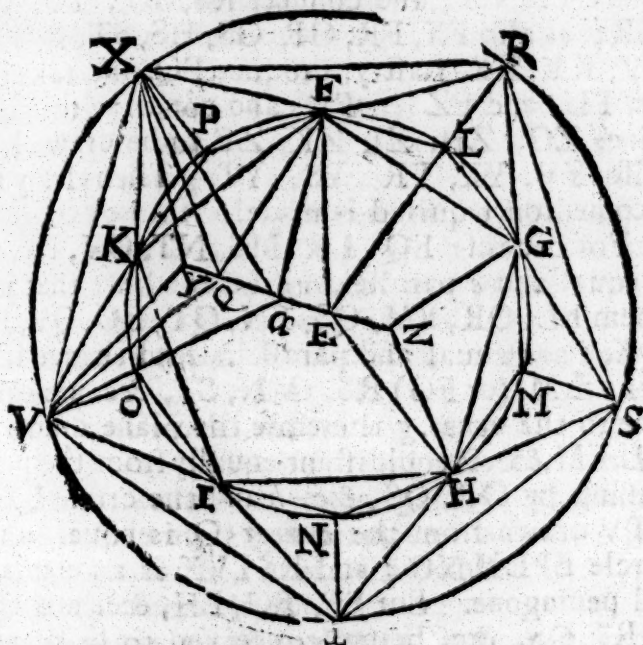
2. The diameter of a sphere containeth in power the side of a tetraedron and of a cube, viz. $ABq\ k = l\ BCq + m\ ACq$.

PROP. XVI.

k 47. 1.

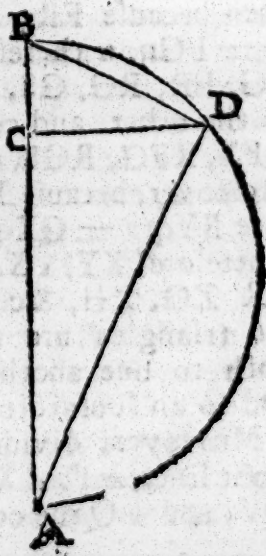
l 13. 15.

m 15. 13.



To describe an Icosaedron ZGHKFTVXRST, and encompass it in the sphere, wherein were contained the foresaid solids; and to demonstrate that FG the side of the Icosaedron is that irrational line, which is called a minor line.

Upon AB the diameter of a sphere describe the semicircle ADB; and make $AB = 5\ BC$. then from C erect CD perpendicular, and draw AD and BD. At the distance $EF = BD$ describe the



a 10. 6.

- b** 11. 4. the circle EPKNG; *b* wherein inscribe the equilateral pentagone FKIHG. Divide equally in two parts the arches FG, GH, &c. and join the right lines FL, LG, &c. being the sides of a decagone. Then *c* erect EQ, LR, MS, NT, OV, PX equal to EF, and perpendicular to the plane FKNG; and connect RS, ST, TV, VX, XR; as also FX, FR, GR, GS, HS, ST, HT, IT, IV, KV, KX. Lastly, produce FQ, and take QY = FL, and EZ = FL and conceive the right lines ZG, ZH, ZI, ZK, ZF to be drawn; as also YV, YX, YR, YS, YT. Then I say the Icosaedron required is made.
- c** 12. 11. For because EQ, LR, MS, NT, OV, PX, are *d* equal and *e* parallel, also those lines that join them EL, QR, EM, QS, EN, QT, EO, QV, EP, QX, *f* are equal and parallel. And thence likewise LM (or FG) RS, MN, ST, &c. are equal one to the other. *g* therefore the plane drawn by EL, EM, &c. is equidistant equally from the plane passing by QR, QS, &c. *h* and the circle QXR-STV drawn from the center Q is equal to the circle EPLMNO; and RSTVX is an equilateral pentagone. But EF, EG, EH, &c. and QX, QR, QS, &c. being conceived to be drawn; then because FRq *k* = FLq + LRq, *l* or EFq *m* = FGq. *n* therefore FR, FG, and so all RS, FG, FR, RG, GS, GH, &c. shall be equal one to the other. and consequently the ten triangles RFX, RFG, RGS, &c. are equilateral and equal.
- d** 13. 1. Moreover, because XQY is a *o* right angle, therefore XYq *p* = QXq + QYq *q* = VXq or FGq. wherefore XY, VX, and likewise YV, YT, YS, YR, ZG, ZH, &c. are equal. Therefore other ten triangles are made, equilateral and equal both to one another, and to the ten former; and so an Icosaedron is made.
- k** 47. 1. **l** 1. **m** 10. 13. **n** 48. 1. **o** 14. 1. **p** 47. 1. **q** 10. 13.

r 15. def. 1. Moreover, divide equally EQ in α , draw the right lines, α F, α X, α V; and because QX r = QV, and α Q the common side, and FQX, EQV are;

are right angle, *f* therefore shall αX be $= \alpha V$; *f* 4. 1.
 and by the same reason all the lines αX , αR , *t* 9. 13.
 αS , αT , αV , αF , αG , αH , αI , αK are equal. *u* 3. 13.
 But because ZQ . QE *t* :: QE . ZE . therefore x 4. 2.
 $Z\alpha q u = 5 E\alpha q x = EQq (EFq) + E\alpha q y = \alpha y$ 47. 1.
 Fq . therefore $Z\alpha = \alpha F$. *z* in like manner $\alpha F = z$ 15. 5.
 $Y\alpha$ therefore the sphere, whose center is α and
 αF the radius, shall pass by the 12 angular
 points of the Icosaedron.

Lastly, because $Z\alpha$, αE :: ZY . QE ; *a* and so *a* 22. 6.
 $Z\alpha q$, $\alpha E q$:: ZYq . $QE q$. *b* therefore $ZYq = 5 b$ 14. 5.
 $QE q$, or $5 BDq = :$ but ABq . BDq *c* :: AB . BC *c* *cor.* 8. 6.
 :: 5. 1. *d* therefore $ZY = AB$. Which was to be *d* 1. *ax.* 1.
done. Therefore If AB be put $\hat{p}$, *e* then $EF = e$ *sch.* 12.
 $\sqrt{ABq}$ shall be also $\hat{p}$. and consequently FG the 10.
 side of the pentagone, and likewise of the Ico- *f* 11. 13.
 saedron, *f* is a minor line. Which was to be *dem.*

Coroll.

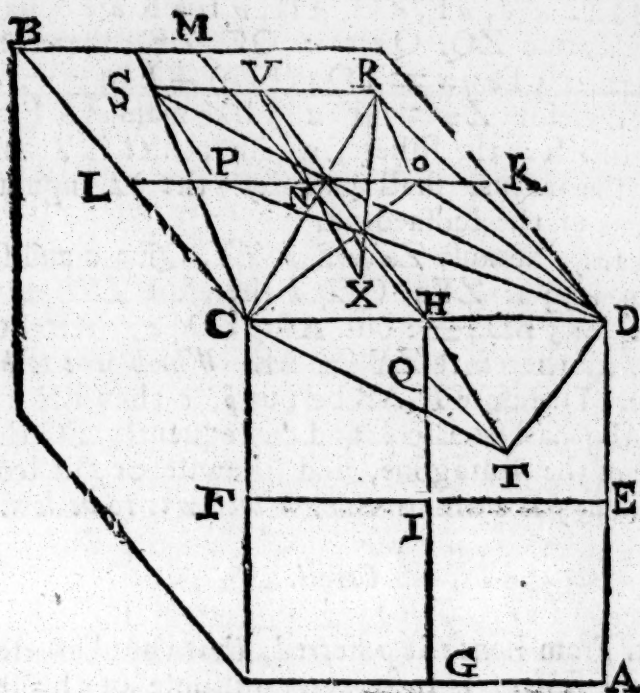
1. From hence is inferred, that the diameter
 of the sphere is in power quintuple of the se-
 midiameter of the circle encompassing the five
 sides of the Icosaedron.

2. Also it is manifest that the diameter of the
 sphere is composed of the side of a hexagone,
 that is, of the semidiameter, and two sides of
 the decagone of a circle encompassing the five
 sides of the Icosaedron.

3. It appears likewise that the opposite sides
 of an Icosaedron, such as RX , HI , are parallels.
 For RX *a* is parallel to LP . *b* parallel to HI .

a 33. 1.
b sch. 26.
 3.

P R O P.



To describe a Dodecaedron, and comprehend it in the sphere wherein the former figures were comprehended; and to demonstrate that the side RS of the Dodecaedron is an irrational line of that sort which is called an apotome or residual line.

a 30. 6. Let AB be a cube inscribed in the given sphere, and let all the sides thereof be divided equally in the point E, H, F, G, K, L, &c. and join the right lines KL, MH, HG, EF. a make $HL:IQ::IQ:QH$; and take NO, NP, = IQ, then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equal to IQ, NO, NP. whence DR, RS, SC, CT, TD, being connected, DRSC shall be a pentagone of the dodecaedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, join

join the right lines DS, DO, DP, CR, CP, HV,
 HT, RX. Because $DOq\ a = DKq\ (b\ KNq) + a$ 47. 1.
 $KOq\ c = 3\ ONq\ (3\ ORq)\ d$ thence $DRq = 4$ b 7. ax. 1.
 $ORq\ e = OPq$, or RSq . therefore $DR = RS$. c 4. 13.
 By the same reason DR, RS, SC, CT, TP are d 47. 1.
 equal. But because $OR\ f$ is $=$ and g parallel e 4. 2.
 to PS , therefore RS, OP , and consequently RS, f const. 96:
 DC shall be also parallels. h therefore these with i i.
 them that conjoin them DK, CS, VH , are in g 33. 1.
 one and the same plane. Moreover, because h 9. 1.
 $HI. IQ\ k :: IQ. (TQ.) QH\ k :: HN. NV$. and k 7. 11.
 both TQ, HN , and $QH, NV\ k$ are perpendicu-
 lar to the same plane, l and so likewise paral- l 6. 11.
 lels, $m\ THV$ shall be a right line. n therefore m 32. 6.
 the Trapezium $DRSC$, and the triangle DTS are n 1. & 2:
 in one plane extended by the right lines DC, TV . 11.
 o therefore $DCTSR$ is a pentagone, and that o 5. 13.
 also equilateral, by what is shewn already. Fur-
 thermore, because $PK. KN :: KN. NP$; and
 $DSq\ p = DPq + PSq\ (PNq) = p\ DKq + PKq$ p 47. 1.
 $+ NPq, q$ thence $DSq = DKq + 3\ KNq = 4$ q 1. ax. 2.
 $DKq\ (4\ DHq)\ r = DCq$. therefore $DS = DC$. and 4. 13.
 whence the triangles DRS, DCT , are equilateral r 4. 2.
 one to another. f therefore the angle $DRS = f$ 8. 1.
 DTC , and likewise the angle $CSR = DCT$.
 therefore the pentagone $DTCSR$ is also equian-
 gular. Moreover, because $AX, DX, CX, \&c.$ are
 semidiameters of the cube, t thence is $XN = IH$ t 15. 13.
 or KN, u and so $XV = KP$; wherefore because u 1. ax. 1:
 RVX , is a x right angle, z thence $RXq = XVq$ x 29. 1.
 $+ RVq\ (NPq) = KPq + NPq\ a = 3\ KNq\ b =$ z 47. 1.
 AXq or $DXq, \&c.$ therefore RX, AX, DX , and
 by the same reason XS, XT, AX , are equal one to
 another. And if by the same method whereby
 the pentagone $DTCSR$ was made, twelve like
 pentagones, touching the twelve sides of the cube,
 be made, they shall compose a Dodecaedron; and
 a sphere passing by their angular points, whose
 radius is AX or RX , shall comprehend that
 Dodecaedron. Which was to be done.

Lastly,

c *constr.*
d 15. 5.

Lastly, because $KN.NO \epsilon :: NO.OK.d$ thence
 $KL.OP :: OP.OK + PL$. Therefore if AB the
diameter of the sphere be supposed $\hat{p}$, then shall

e 15. 13.
f *schol.* 12.
10.
g 6. 13.

$KL \epsilon = \sqrt{\frac{AB}{3}} f$ be also $\hat{p}. g$ whence OP or RS
the side of the dodecaedron shall be a residual
line. Which was to be demonstrated.

Coroll.

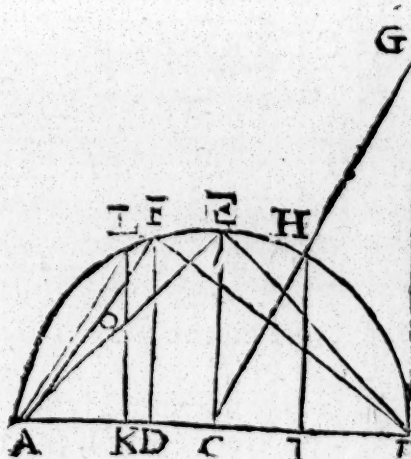
From this demonstration it follows, 1. that if
the side of a cube be cut in extrem and mean
proportion, the greater segment shall be the side
of the dodecaedron described in the same sphere.

2. If the lesser segment of a right line, cut
in extrem and mean proportion, be the side of
the dodecaedron, the greater segment shall be
the side of the cube inscribed in the same sphere.

3. It is manifest also, that the side of the cube
is equal to the right line which subtendeth the
angle of a pentagone of the dodecaedron inscri-
bed in the same sphere.

P R O P. XVIII.

a 10. 1.
b 10. 6.



To find out the
sides of the precedent
five figures, and com-
pare them together.

Let AB be the
diameter of the
sphere given, and
 AEB the semicir-
cle, and let AC be
 $a = \frac{2}{3} AB$, and AD
 $b = \frac{1}{3} AB$. then e-
rect the perpendi-
culars CE, DF , and

and $BG = AB$. join AF, AE, BE, BF, CG ;
and let fall the perpendicular HI from H
and

and CK being taken equal to CI, from K erect the perpendicular KL, and join AL. Lastly, c make AF. AO :: AO. OF.

Therefore 3. 2 d :: AB. BD e :: ABq. BFq the side of a Tetraedron and 2. 1 :: a AB. AC :: ABq. BEq f the side of an Octaedron.

Also 3. 1 d :: AB. AD e :: ABq. AFq, g the side of an Hexaedron.

Moreover, because AF. AO h :: AO. OF. k thence shall AO be the side of a Dodecaedron.

Lastly, BG, (2 BC.) BC l :: HI. IC. m therefore HI = 2 CI n = KI. therefore HIq o = 4 CIq. consequently CHq p = 5 CIq q therefore ABq = 5 KIq. a therefore KI or HI is a radius of a circle enclosing the pentagone of an Icosaedron; and AK or IB r is the side of a decagone inscribed in the same circle. f whence AL shall be the side of a pentagone, t and also the side of an Icosaedron. Whereby it appears that BF, f 10. 13.

BE, AE are p q. and AL, AO p q, and BF t 16. 13. BE, and BE AF, and AF AO. And because u 1. 6. 3 AFq = ABq u = 5 KLq, and AF x AO AF x OF, x and so AF x AO + AF x OF AF x y 1. 2. OF, y that is, AFq AFq z 2 AOq. a thence shall 3 AFq (5 KLq) be AFq z 2 AOq. consequently KL a 47. 1. AFq and much rather AL AF.

That we may express these sides in numbers; If AB be supposed $\sqrt{60}$, then, reducing what is already shewn to supputation, $BF = \sqrt{40}$, and $BE = \sqrt{30}$, and $AF = \sqrt{20}$. Also $AL = \sqrt{30} - \sqrt{180}$ (for $AK = \sqrt{15} - \sqrt{3}$, and $KL(HI) = \sqrt{12}$.) Lastly $AO = \sqrt{30} - \sqrt{500} (\sqrt{25} - \sqrt{5}.)$

Schol.

Schol.

It is very apparent that besides the five aforesaid figures, there cannot be described any other regular solid figure (viz. such as may be contained under ordinate and equal. plane figures.)

a 21. 11.

b See sch.

32. 1.

For three plane angles at least are required to the constituting of a solid angle; *a* all which must be less than four right angles. *b* But 6 angles of an equilateral triangle, 4 of a square, and six of a hexagon, do severally equal 4 right angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagone, &c. do exceed 4 right angles; Therefore only of 3, 4, or 5 equilateral triangles, of 3 squares, or 3 pentagones; it is possible to make a solid angle. Wherefore besides the five above mentioned, there cannot be any other regular bodies.

Out of P. Herigon.

The Proportions of the sphere and the five regular figures inscribed in the same.

Let the diameter of the sphere be 2. then shall

The Peripherie or circumference of the greater circle, be 6. 28318.

The superficies of the greater circle, 3. 14159.

The superficies of the sphere, 12. 56637.

The solidity of the sphere, 4. 1879.

The side of the tetraedron, 4. 62299.

The

EUCLIDE'S *Elements*:

337

The superficies of the tetraedron, 4, 6188.

The solidity of the tetraedron, 0, 15132.

The side of the hexaedron. 1, 1547.

The superficies of the Hexaedron, 8.

The solidity of the hexaedron, 1, 5396.

The side of the Octaedron, 1, 41421.

The superficies of the Octaedron, 6, 9282.

The solidity of the Octaedron, 1, 33333.

The side of the Dodecaedron, 0, 71364.

The superficies of the Dodecaedron, 10, 51462.

The solidity of the dodecaedron, 2, 78516.

The side of the Icosaedron, 1, 05146.

The superficies of the Icosaedron, 9, 57454.

The solidity of the Icosaedron, 2, 53615.

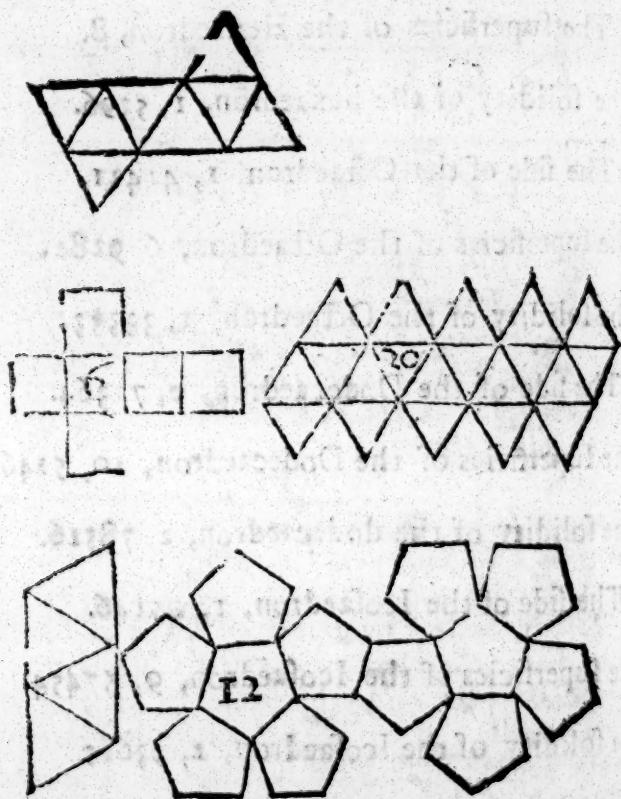
Y

If

The

The thirteenth Book of, &c.

If five equilateral and equiangular figures, like these in the schemes beneath, be made of Paper, and rightly folded, they will represent the five regular bodies.



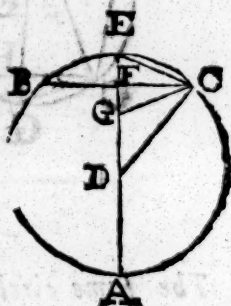
The end of the thirteenth Book.

THE

The FOURTEENTH BOOK OF EUCLIDE'S ELEMENTS.

PROPOSITION I.

A Perpendicular line DF drawn from D the center of a circle ABC to BC the side of a pentagon inscribed in the said circle, is the half of these two lines taken together, viz. of the side of the hexagon DE , and the side of the decagon EC inscribed in the same circle ABC .



Take $FG = FE$, and draw CG : *a* Then CE is $= CG$. therefore the angle CGE *b* $= CEG$ *b* $= ECD$. therefore the angle ECG *c* $= EDC$. *d* $= \frac{1}{4} ADC$ *e* $= \frac{1}{4} CED$ ($\frac{1}{2} ECD$.) *f* consequently the angle $GCD = ECG = EDC$. *g* wherefore $DG = GC$ (*CE*.) therefore $DF = CE$ (DG) $+ EF = DE + CE$.

Which was to be dem.

2

PROP. II.

If two right lines AB, DE , $A \quad G \quad B \quad C$
be cut according to extrem ——— 1 — 1 ———
and mean proportion ($AB. D \quad H \quad E \quad F$
 $AG :: AG. GB.$ and $DE. ——— 1 — 1 ———$
 $DH :: DH. HE.$) they shall
be cut after the same manner, viz. into the same
proportions ($AG. GB :: DH. HE.$)

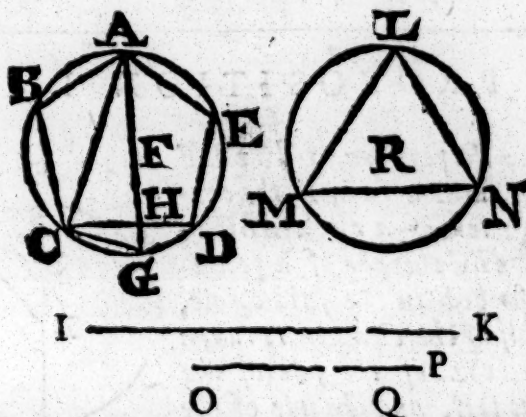
Y 2

Take

a 4. 1.
b 5. 1.
c 32. 1.
d hyp. and
33. 6.
e 20. 3.
f 7. *ass.*
g 6. 1.

Take $BC = BG$; and $EF = EH$. Then $AB \times BG$
 is $a = 5$ AGq . wherefore $ACq \ b = 4$ $ABG + AGq$
 $c = BGq$. In like manner shall $DFq \ b = 5$ DHq .
 d therefore $AC. AG :: DF. DH$. whence by ad-
 dition $AC + AG. AG :: DF + DH. DH$. that is,
 $2 AB. AG :: 2 DE. DH$. e consequently $AB. AG$
 $:: DE. DH$; f whence by division $AG. GB :: DH.$
 HE . Which was to be demonstrated.

P R O P. III.



a sch. 47. I.

b 30. 6.

c 47. I.

d 4. 2.

e 10. 13.

f 1, and

3. ax.

g 8. 13.

h 2. 13. &

i 16. 5.

k 22. 6. &

l 4. 4.

m 15. 13.

n const.

o cor. 16.

p 13.

q 12. 13.

r 10. 13.

s 15. 5.

* before

t 1. ax. 1.

u and schol.

v 48. 1.

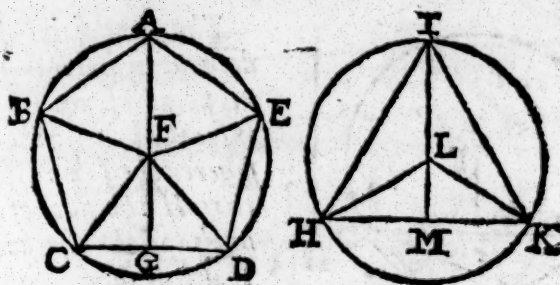
w 1. def. 3.

The same circle ABD comprehends both $ABCDE$ the pentagone of a Dodecaedron, and LMN the triangle of an Icosaedron inscribed in the same sphere.

Draw the diameter AG , and the right lines AC , CG . and let IK be the diameter of the sphere, a and $IKq = 5$ OPq . b and make $OP. OQ :: OQ. QP$. Because $ACq + CGq = AGq$ $d = 4$ FGq ; and $ABq = FGq$. f thence $ACq + ABq = 5$ FGq . moreover, because $CA. AB \ g :: AB. CA - AB$; and $OP. OQ :: OQ. QP$. h and so $CA. OP :: AB. OQ$. k therefore 3 ACq (l IKq) 5 OPq (m IKq) $:: 3$ ABq . 5 OQq . therefore 3 $ABq = 5$ OQq . But because $ML \ n$ is the side of a pentagone inscribed in a circle, whose radius is OP , thence 15 RMq . $o = 5$ MLq $p = 5$ OPq . q 5 $OQq = 3$ $ACq + 3$ ABq $q = 15$ FGq . r therefore $RM = FG$. s and consequently the circle ABD is $=$ to the circle LMN . Which was to be demonstrated.

P R O P.

PROP. IV.



If from *F* the center of a circle encompassing the pentagon of a dodecaedron *ABCDE*, a perpendicular line *FG* be drawn to one side of the pentagon *CD*; the rectangle contained under the said side *CD* and the perpendicular *FG*, being thirty times taken, is equal to the superficies of a Dodecaedron. Also,

If from the center *L* of a circle inclosing the triangle of an Icosaedron *HIK*, a perpendicular line *LM* be drawn to one side of the triangle *HK*, the rectangle contained under the said side *HK* and the perpendicular *LM*, being thirty times taken, shall be equal to the superficies of an Icosaedron.

Draw *FA, FB, FC, FD, FE*. a then shall the triangles *CFD, DFE, EFA, AFB, BFC* be equal. but $CD \times FG$ b = 2 triangles *CFD*. therefore 30 $CD \times GF$ c = 60 *CFD* d = 12 pentagones *ABCDE* e = to the superficies of a dodecaedron. Which was to be demonstrated.

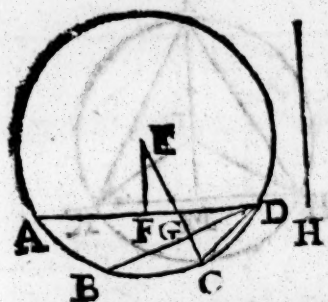
a 8. 1.
b 4. 1.
c 15. 5.
d 6. ax.
e 17. 3.
f 41. 1.
g 15. 5.
h 16. 13.

Draw *LI, LH, LK*; then $HK \times LM$ f is = 2 triangles *LHK*. therefore 30 $HK \times LM$ g = 60 *HLK* h = 20 *HIK* b = to the superficies of an Icosaedron. Which was to be demonstrated.

Coroll.

$CD \times FG. HK \times LM$ k :: the superficies of a dodecaedron to the superficies of an Icosaedron. k 15. 5

PROP. V.



The superficies of a Dodecaedron hath to the superficies of an Icosaedron inscribed in the same sphere, the same proportion that H the side of a cube hath to AD the side of an Icosaedron.

Let the circle $ABCD$

a 3. 14.

enclose both the pentagone of a dodecaedron, and the triangle of an Icosaedron; whose sides are BD , AD , upon which from the center E let fall the perpendiculars EF , EG ; and draw CD .

b 9. 13.

c 1. 14.

d cor. 12.

13.

e 15. 5.

f cor. 17.

13.

g 2. 14.

h 1. 6.

k 7. 5.

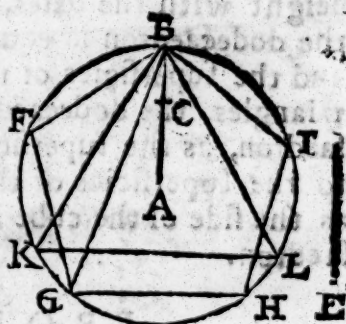
l cor. 4. 14.

Because $EC + CD$. EC $b :: EC$. CD , thence EG . (c , $EC + CD$) EF (d , $\frac{1}{2} EC$) $e :: EF$. EG — EF . (i , CD) but H . BD $f :: BD$. H — BD . g therefore H . $BD :: EG$. EF . consequently $H \times EF = BD \times EG$. wherefore since H . AD $h :: H \times EF$. $AD \times EF$. k thence shall be H . $AD :: BD \times EG$. $AD \times EF$ $l ::$ the superficies of a dodecaedron to the superficies of an Icosaedron. Which was to be demonstrated.

PROP

PROP. VI.

If a right line AB be cut in extrem and mean proportion, then as the right line BF , containing in power that which is made of the whole line AB , and that which is made of the greater segment AC , is to the right line E containing in power



or that which is made of the whole line AB , and that which is made of the lesser segment BC ; so is the side of the cube BG to the side of an Icosaedron BK inscribed in the same sphere with the cube.

In the circle, whose semidiameter is AB , inscribe $BFGHI$ the pentagone of a dodecaedron, and BKL the triangle of an Icosaedron, a wherefore BG shall be the side of a cube inscribed in the same sphere. therefore BKq $b = 3$ ABq ; and $Eqc = 3$ ACq . therefore BKq . $Eqd :: ABq$. $ACqe :: BGq$. BFq . wherefore by inversion BGq . $FKq :: BFq$. Eqf whence BG . $BK :: BF$. E . Which was to be demonstrated.

a cor. 17.
13.
b 12. 13.
c 4. 13.
d 15. 5.
e 2. 14.
f 22. 6.

PROP. VII.

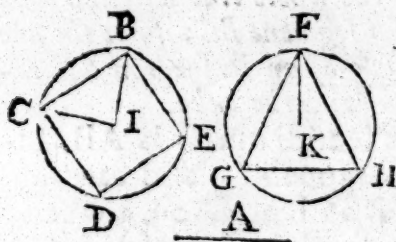
A Dodecaedron is to an Icosaedron, as the side of a Cube is to the side of an Icosaedron, inscribed in one and the same sphere.

Because a the same circle comprehends both the pentagone of a dodecaedron, and the triangle of an Icosaedron, b the perpendiculars drawn from the center of the sphere to the planes of the pentagone and triangle, shall be equal one to another. Therefore if the dodecaedron and Icosaedron be conceived divided into pyramids, right lines being drawn from the center of the sphere

a 3. 14.
b 47. 1.

- c* 5. and 6. 12. Wherefore since the pyramides *c* of equal height with the bases, and the superficies of the dodecaedron is equal to twelve pentagones, and the superficies of the Icosaedron to twenty triangles, the dodecaedron shall be to the Icosaedron, as the superficies of the dodecaedron is to the superficies of the Icosaedron, *d* that is, as the side of the cube is to the side of the Icosaedron.
- d* 5. 14.

P R O P. VIII.



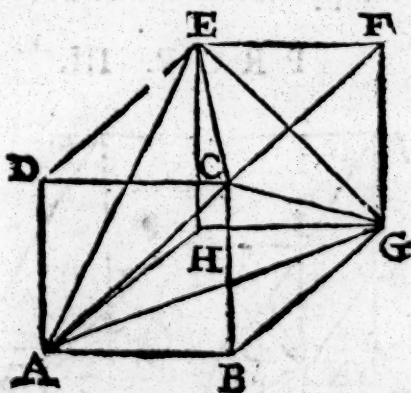
The same circle *BCDE* comprehends both the square of the cube *BCDE*; and the triangle of the octaedron *FGH* inscribed in one and the same sphere.

- Let *A* be the diameter of the sphere. Because *Aq* *a* = 3 *BCq* *b* = 6 *BIq*; and also *Aq* *c* = 2 *GFq* *d* = 6 *KFq*; thence shall *BI* be = *KF*. *e* therefore the circle *CBED* = *GFH*. Which was to be demonstrated.
- a* 15. 13.
b 47. 1.
c 14. 13.
d 22. 13.
e 2. def. 3.

The End of the fourteenth Book.

The FIFTEENTH BOOK
OF
E U C L I D E'S
E L E M E N T S.

PROPOSITION I.



IN a cube given *ABGHDCFE* to describe a pyramid *AGEC*.

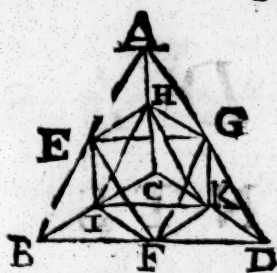
From the angle *C* draw the diameters *CA*, *CG*, *CE*; and connect them with the diameters *AG*, *GE*, *EA*. All which are equal among themselves, as being the diameters of equal squares: therefore the triangles *CAG*, *CGE*, *CEA*, *EAG* are equilateral and equal; and consequently *AGEC* is a pyramid, which insits upon the angles of the cube, and therefore *p* is inscribed in it. Which was to be done. a 47. 1.

PROP. II.

b 31. def.

PROP II.

In a pyramide given $ABDC$ to describe an octaedron EGR . IFH .



a 10. 1.

b 4. 11.

c 27. def.

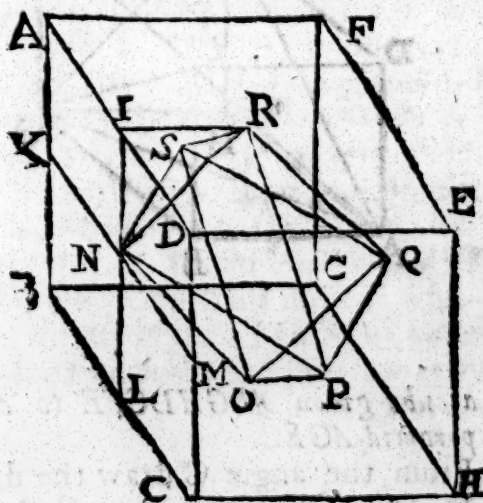
25.

d 31. def.

11.

a Bisect the sides of the pyramide in the points, E, I, F, K, G, H. which join with the right lines EF, FG GE, &c. All these are *b* equal one to the other; consequently the 8 triangles EHI, IHK, &c. are equilateral and equal, and so make *c* an octaedron described *d* in the given pyramide. Which was to be done.

PROP. III.



In a cube given $CHGBDEFA$ to describe an octaedron $NPQSOR$.

* 8. 4.

a 4. 1.

b 31 and

27. def. II.

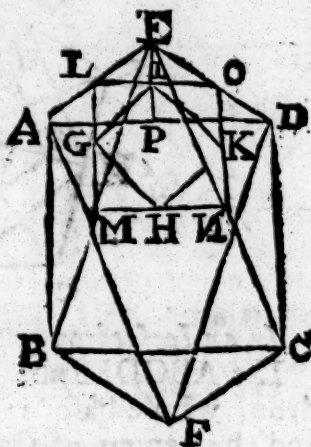
Connect * the centers of the squares N, P, Q, S, O, R with the twelve right lines NP, PQ, QS, &c. which are *a* equal among themselves; and so make 8 equilateral and equal triangles: wherefore *b* the Octaedron $NPQSOR$ *b* is inscribed in the cube. Which was to be done.

PROP.

P R O P. IV.

In an Octaedron given
ABCDEF, to inscribe a
 cube.

Let the sides of the
 pyramide *EABCD*,
 whose base is the
 square *ABCD*, be equal-
 ly bisected by the right
 lines, *LM*, *MN*, *NO*,
OL, which are *a* equal
 and *b* parallel to the
 sides of the square *AB*-
CD. *c* then the quadri-



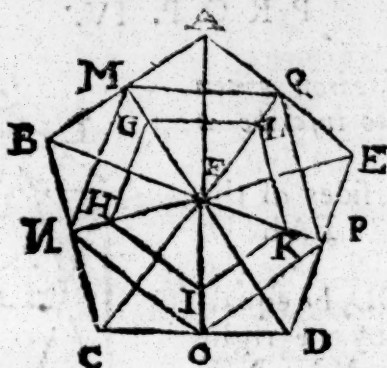
a 4. 1.
b 2. 6.
c 29. def. 1.

lateral *LMNO* is a square. In like manner, if
 the sides of the square *LMNO* be equally bi-
 sected in the points *G*, *H*, *K*, *I*, and *GH*, *HK*,
KI, *IG* connected, *GHKI* shall be a square.
 And if in the other 3 pyramides of the octae-
 dron, the centers of the triangles be in the
 same sort conjoined with right lines, then other
 squares will be described like and equal to the
 square *GHKI*. wherefore six such squares shall
 make a cube, which shall be described within
 an octaedron, *d* being its eight angles touch the
 eight bases of the octaedron in their centers.
 Which was to be done.

d 31. def.
 11.

PROP.

PROP V.

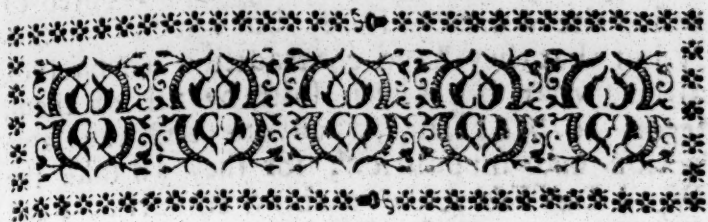


In an Icosaedron given to inscribe a Dodecaedron.

Let ABCDEF be a pyramide of the Icosaedron, whose base is the pentagone ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL shall be a pentagone of the dodecaedron to be inscribed.

For the right lines, FM, FN, FO, FP, FQ, passing by the centers of the triangles, *a* do equally divide their bases into two parts. *b* therefore the right lines MN, NO, OP, PQ, QM *c* are equal one to the other; *d* whence also the angles MFN, NFO, OFP, PFQ, QFM are equal. therefore the pentagone GHIKL is equiangular. *e* and consequently equilateral, being FG, FH, FI, FK, FL *f* are equal. And if in the other eleven pyramids of the Icosaedron, the centers of the triangles be in like sort conjoined with right lines, then will pentagones equal and like to the pentagone GHIKL be described. Wherefore 12 of such pentagones shall constitute a dodecaedron; which also shall be described in the Icosaedron, seeing the twenty angles of the dodecaedron consist upon the centers of the twenty bases of the Icosaedron. Whereby it appears that we have described a dodecaedron in an Icosaedron given. *Which was to be done.*

F I N I S.



EUCLIDE'S DATA.

A

*Commentary or Preface written by
the Philosopher MARINUS, on
EUCLIDE'S DATA.*

IN the first place we ought to set down (as a Foundation) what that is, which we call *DATUM* or GIVEN; then to consider the Profit and Utility thereof; and in the third place, to what Art or Science this Tract doth appertain.

The Word *DATUM* therefore is diversly defined, for the Antients have defined it after one manner, and later Writers after another, whence it follows that it seemeth a difficult thing to give a true Explication thereof; for some of them have not delivered the Definition of the Word, but have with much Labour and Trouble sought certain Proprieties thereof, and some others collecting and mingling what have been delivered

delivered by others before, have endeavoured to define the Word *DATUM*, but not so exquisitely but that they have contradicted themselves, altho' what have been said by all of them, seems to be grounded on one and the same notion and supposition; for they all take the Word *DATUM* to be a thing comprised; and therefore among such as have endeavoured to describe it most simply, and with some simple difference, some of them have taken the Word *DATUM* to be the same with *ORDINATUM*, and so *Apollonius* understands it in his Tract of Inclinations, and in his universal Tract; and so others, as *Diodorus* takes it to be *COGNITUM KNOWN*, for in this Signification he takes the right line and the angles to be given, and all that may arrive to our Knowledge, altho' we may not be able well to express it. But others have believed that it hath the same signification as the word [*Effabile*] that may be declared, and so *Ptolomy* would have it, who calls those things *GIVEN*, whose measure is known whether precisely, or near the matter. Others have also thought the Word *DATUM* to be what is granted us by the proposer in the Hypothesis, being that in the first Elements, a point given, and a right line given, is diversly taken (that is to say, that who so would give and determine the quantity of a right line) all which things signify some *COMPREHENSION*; and therefore of all these Definitions, those are most agreeable, which do most openly declare the *COMPREHENSION*, as we shall make evident by what follows.

Let us now unfold the diverse Opinions of those, who writing the nature of *DATUM GIVEN*, have not taken one simple Mark, or only Character for its Definition; and let us reduce it as in a Summary or Epitomy, to the end

and we may with the more ease know or number all their Differences. Some of them then have defined *DATUM* to be *Ordinatum* and *Porimon* together, and others *Ordinatum* and *Cognitum* together, and others *Porimon* and *Cognitum* together: Wherefore all seem to have so defined it as to have had regard to the *Comprehension*, or *Assuming* and *Invention* of the thing given; and to the end that we may the better conceive their Opinions, and that from the saying of many we may be able to draw a true Definition of what is proposed, we will take notice in the first place of the Signification of all the simple Terms which they make use of, as also of the Terms opposed to them, to wit, *Inordinatum* and *Incognitum*, *Aporon* and *Irrational*; for those things appertain to this Geometrical Business, to natural things, and to Mathematical Discipline.

Now we may call that *Ordinatum* (or Regulated) which doth always keep and observe that for which it is said to be ordered, whether you regard its Magnitude or Species, or touching some other such like thing: It is also thus defined, *Ordinatum* is that which cannot be done in divers manners, but in one only manner, and in some determined place: As for example, A right line drawn by two given points, is said to be ordered, by reason it cannot be otherwise done, nor in divers manners. But an angle passing by two points is said to be *Inordinatum* (or disordinate and irregular) for that it is made in infinite and divers manners by a great or small circle described by two points *ad infinitum*. Contrariwise, an angle constituted by three points, is said to be *Ordinatum*, as also those things which follow are said to be *Ordinatum*, as to constitute an equilateral triangle on a right line, for it cannot be diversly made, but unchangeably, on both the given

given extremities of the line. Again, to divide a given right line according to a given proportion, for that cannot be done but in one certain point. The things *Inordinatum* are such as are done contrary to those last mentioned, as to constitute a Scalene triangle, and to divide a right line indefinitely. Wherefore the Problem is ordered, is proposed in the determination, considering that a certain thing may be in one manner said to be *Ordinatum*, and *Inordinatum* in another, as an equilateral triangle, considering the equality of the sides, it is *Ordinatum*, but considering its Magnitude it is *Inordinatum*, being in no wise determined.

But we call that *Cognitum* which is notorious, as clear and comprehended of us, and *Incognitum* that which is not known, or comprehended of us, as the length of a way is said to be known, when we know how many Miles it contains; also that the three lines of a Rectiline triangle are equal to two right angles, and in like manner that the Binomial is irrational, such things are known, as also that it is only one right line that can touch a spiral line from a given point without it, from both parts; for if there were yet another line, two right lines would inclose a space, which is impossible. Again, irrational things are not said to be unknown, but such of them only which are neither known nor comprehended of us.

Porimon is that which we may make and constitute, that is to say, bring to our Understanding. Again, it is defined thus, *Porimon* is that which may be exhibited by Demonstration, or which is apparent without Demonstration, as to describe a circle from a center and with a space, as also to constitute not only an equilateral triangle, but also a Scalene; or to find a *Binomium*, or to find two right lines rational, commensurable in power only, and other things

things which are known infinitely are *Porimon*; as to describe a circle by two points.

Aporon is wholly opposite to *Porimon*, as for example, the Quadrature of a circle, for that it hath not as yet been found; altho' it be certainly known that it may be: Nevertheless the manner of finding it out hath not been to this present comprehended. But we speak here of that which is already known, which is called *Porimon principale*; for what hath not been as yet made, and yet nevertheless is possible, is called *Poriston*, (or feasible) altho' the Construction be yet unknown. But *Aporon*, as hath been afore said, is opposite to *Porimon*, and is that whose Nature is not as yet decided, nor well determined.

Effabile, that is to say, rational (or speakable and explicable) is that whose Magnitude, Species, and Position, we may be able to declare; but this Definition is a little too general, for properly, and according to it self, *Effabile* is that which is known by certain things, and according to a Measure given by Position, as of a span, or a finger's breadth, &c.

These things then being thus unfolded, we may easily perceive in what all those things that we have afore spoken of do agree together, and wherein they do differ; and first of all how *Ordinatum* and *Cognitum* do agree together, and likewise their opposites the one to the other, for it cannot be said that any one of those things by counterchanging is the other, nor yet that the one hath not more extent than the other, altho' they agree in many things, as to describe a right line by two points, and to constitute an equilateral triangle by three circles; but to square a circle, that is indeed *Ordinatum*, yet nevertheless *Incognitum*. Also that at a point of a spiral line there is but one touch line, that is, of the kind we call *Ordinatum*, and cannot
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otherwise done, yet nevertheless the Demonstration and Construction thereof is not yet known. Again, the indefinite section, and the Construction of the Scalenum is *Cognitum*; but is not *Ordinatum*; insomuch as it is manifest, that amongst those things which are *Ordinatum*, there are some that are *Cognitum*, and others that are *Incognitum*; and contrariwise, that amongst the things that are *Cognitum*, there are some that are *Ordinatum*, and others that are *Inordinatum*; and therefore those things answer one another, as among Living Creatures, that which hath reason with that which hath Feet, for there is no quality amongst them, neither doth the one extend more than the other.

In like manner, *Ordinatum* and *Inordinatum* agree together, respecting *Porimon* and *Aporon*; seeing that between them there is a very great Resemblance, and because that they do differ only in the manner before expressed; for in truth the spiral line is *Ordinatum*; but it was not *Porimon* before *Archimedes*; and by the same reason those things that are inordinate and known by an infinity of ways and means are *Porimon*, if any one should undertake to invent their Constitution and Construction. Yet nevertheless they are not ordinate, as to constitute a Scalenum triangle, it being no difficult thing to make known the constitution thereof by an equilateral triangle, yet it is most easy, altho' it be inordinate and known by an infinity of ways.

And in the same manner do agree *Ordinatum* and *Inordinatum*, together with *Effabile* and *Irrationale*; for they agree together in many things, differing nevertheless by the foregoing reason, seeing those things there mentioned bear no equality to each other, neither doth one thing contain the other; for all Binomiums and such as are taken as Irrationals, are indeed ordinate; but

but yet they are not *Effabile*, or expressible, or to be unfolded, as the diameter of the square is in respect of its side. Now touching *Effabile*, there are divers inordinate; because they are diversly known, and indeterminately; for a Scalenum triangle may be measured by a defined and proposed measure, as explicable, altho' it be inordinate.

Now it is easy to see the agreement that there is between *Cognitum* and *Porimon*, but it is a difficult thing to express or unfold their difference; forasmuch as in their Natures they come so near to one another, as that there seems to be an equality between them: Nevertheless, there will some difference appear to him that shall consider it more strictly; for let it be consented to, that there can be only one line that can touch a spiral line in a certain point that is *Cognitum*, yet notwithstanding the Problem is not *Porimon*, it being not as yet comprehended; so as that all that which is *Cognitum*, is not therefore *Porimon*. But all that which is *Porimon* is also *Cognitum*, and therefore *Cognitum* appears to be of a greater extent than *Porimon*.

Now *Cognitum*, *Porimon*, and *Effabile*, do agree in some certain things, and do differ in other things by the same reason before alledged; for those lines which are there called Irrationals are in truth known; and yet nevertheless are not *Effabile* or explicable. Contrariwise, every number is indeed *Effabile*, and yet every number is not *Cognitum*. But *Effabile* is always of its own nature expressible, altho' that some lengths may be now *Effabile*, and at another time not; if it be examined with some other according to one and the same measure. But also that same length is sometimes known, and other times not, tho' they wholly agree with one another. Now it is a difficult thing to find something that shall be *Effabile* and *Incognitum*, for *Cognitum*

seems to be of a greater extent than *Effabile*, and by those things it is manifest that *Porimon* and *Aporon* do differ from RATIONAL or *Effabile*, and from IRRATIONAL; for of IRRATIONAL some of them may be *Porimon*, but of RATIONAL none of them can be Irrationals; and therefore it is very easy to perceive in what the before expressed things agree. Notwithstanding they seem to agree together, in such sort as that *Porimon* seems to be of a greater extent than *Effabile*.

Now by these things we may come to know the difference of those things that have been before spoken of, for in truth *Effabile* and *Irrationale* are so termed in respect of measure, which notwithstanding is not as yet arrived to our Understanding, seeing that something that is rational, may be as yet unknown to us, and in like manner may be rational, and yet may never be comprehended so to be. But *Ordinatum* and *Inordinatum* is so termed according to it self, and according to the proper nature of the thing on which we contemplate, altho' it be not comprehended by us. As *Archimedes* had perceived some things to be ordinate from the nature of the things, the which *Serenus* had before contemplated. But *Cognitum* and *Incognitum* is spoken in respect of us, so as the things before mentioned do differ among themselves; for these have respect to us, the others, some of them to their proper nature, and the rest to measure.

Having then explained the agreements and differences of the things that have been proposed, it remains now that we consider what is meant by the Word *DATUM*. for of all those that believe the Word *DATUM* to be that which is consented to by the Proposer in the Hypothesis, are wide from what is sought; because that all the Elements of the things GIVEN are not com-

posed

posed of this sort of GIVEN, which is according to the Hypothesis, as may be seen in those Tracts which have been made on this Subject GIVEN. Wherefore waving this Opinion, let us judge of the Definitions of others.

Then that which is consented to, or granted in the Hypothesis, is something which is consequently known by the Principles; but such as make use of Definitions of one only Word, do define it and remark it by some one of the before mentioned, as hath been spoken in the beginning, so as that almost all seem to have had this common notion of GIVEN. to wit, that it is comprehended even as the Word *DATUM* doth also manifest it to be; and amongst those, these are the chief that do define it by the Hypothesis or Supposition; and others have had regard to what is consented to or granted. But we making use of the said things as a Rule and Direction to judge rightly, we may be able to find out a perfect Definition of *DATUM*; for it is certain that it ought to equal and be convertible with the thing defined, which is one thing proper to good Definitions. Now such seems to be the Definition of the thing proposed, which among the most simple and plain Expositors is defined *Porimon*, and among the more acute, that which defineth it to be *Porimon* and *Cognitum* together; but all the rest are imperfect; for that which defineth it *Ordinatum* is not sufficient for the Comprehension and Knowledge of *DATUM*; because that neither wholly ordinate nor alone ordinate, is not comprised, seeing that there are things inordinate that have the same Condition, as hath been shewn. Again, that reason gives not Satisfaction neither, which describes it to be *Cognitum*; for all that is known is not comprehended, altho' that alone *Cognitum* be comprehended. Moreover, that also is not perfect which defineth it to be *Effabile*, for *Effabile*

is not alone comprehended ; seeing that some of the Irrationals are also comprehended. In like manner, all *Effabile* is not comprehended, as hath been before declared. Now amongst the Definitions which expound it by the only Word, there remains that which defineth it to be *Porimon*, which seemeth greatly to manifest the Comprehension ; for whole *Porimon* and alone *Porimon* is comprehended. Wherefore *EUCLIDE* himself useth such a Definition in a Description of all the kinds of GIVEN by him conceived and regarded. But amongst such Definitions as are compounded, that is a perfect Definition which defineth *DATUM* to be *Cognitum* and *Porimon* together, having *Cognitum* for analogical kind, and *Porimon* for difference ; but that is imperfect which hath *Ordinatum* and *Porimon* together ; for those things which are such, are not alone GIVEN, and that which defineth it *Ordinatum* and *Effabile* together, comprehendeth likewise the GIVEN, with the defect or want. But that of *Cognitum* and *Ordinatum* together, is not to be received or admitted, because it doth exceed what is defined, for such is not given alone : Therefore those only which have declared that *DATUM* is *Cognitum* and *Porimon* together, seem to have attained the notion of GIVEN, for that which is such is all, and alone comprehended, which two things ought to be in those Definitions that are well given. But the former comes near to those which have thus defined it ; *DATUM* is that to which we may find an equal, according to those things we have proposed in the first Principles and Hypothesis, of which number *EUCLIDE* is one making use throughout of the Word *παισάδα*, which signifies to exhibit or invent, altho' he leaves *Cognitum* as a Consequent of *Porimon* ; some one nevertheless might reprove him, for that in the first place he hath not defined *DATUM* in general ; but immediately

diately some of the kinds of GIVEN, altho' in his Elements of GEOMETRY, he hath defined the line simple before the Species or Kinds.

What is the Utility and Profit that ariseth from this Tract of DATA, or things GIVEN.

After having explained universally, and according to what seemed necessary for our present Use, what this Word *DATUM* signifieth: It follows to shew the Utility of this Tract. Now this Tract is such, as that it is not only ordained and instituted for its own respect, but for some other thing; for it is very necessary to a place which is called Resolved, and we have already declared elsewhere how much Strength a resolved place doth obtain in Mathematical Disciplines, as also in Opticks and Cannons, which come very near to them, as well for that Resolve is an Invention of the Demonstration, as for that in such like things it serves us much for the Invention of the Demonstration, or for that it is much more excellent to meet with a Resolutive power, than to enjoy divers particular Demonstrations.

To what Art or Science this Tract is referred.

NOW seeing the Consideration of GIVEN is useful and profitable in all these kinds of Arts, for that it serveth much to RESOLUTION, it may well be said to be recalled, not only to one only Science, but to the Mathematicks universally which treat of Numbers, Time, Swiftnes, and such like things, which treateth likewise of Reasons, as also of Proportions, and in a word of all *Medietez*: Wherefore for the perfect and demonstrative Knowledge of things

GIVEN, being of so great Utility, *EUCLIDE* hath taken pains to frame this Book of things GIVEN, which Author amongst all such as have composed the Elements of Geometry, hath justly deserved the first place and rank, and who having invented the Elements, or rather the Introductions almost of all Mathematical Disciplines, to wit, of all Geometry in 13 Books, of Astronomy, of Musick, and Opticks, he hath left in Writing the Elements RESOLUTIVE, in this Treatise of things GIVEN; but as he was a Geometrician, he hath particularly accommodated to Magnitudes, that was of the GIVEN, yet nevertheless common in other things, which Method hath also been observed by him, when treating universally of Reasons and Proportions, he appropriates them to the Magnitudes mentioned in this Fifth Book of Planes.

Now it hath been declared in general what is the meaning of *DATUM*, to what Science it appertaineth, and how profitable the Contemplation thereof is. We will add to what hath been said, the Description of this Science which treats of things GIVEN; seeing that it is (as appears by what hath been said) a Comprehension in all manner of things GIVEN; and of their Accidents and Proprieties. But having respect to the proposed Book, we shall declare it to be an Elementary Doctrine of the whole Knowledge of things GIVEN, whence it follows that it will be very profitable, as also the things therein contained, seeing they refer to things GIVEN.

Now this Book is divided according to the Species or Kinds of the things GIVEN, and in the first Section are contained those things which are given by Reason. Secondly, such as are given by Position: and lastly, such as are given by Species or Kind; for that which is given by Magnitude is simple, and particularly contained in the

the others, and principally in those things given by Species or Kind. Now he hath begun with those things given by Reason and Position, forasmuch as those that are given by Species are constituted of them. *EUCLIDE* gives yet another Division to this Book, for that he divides it into universal Magnitudes, Lines and Superficies, and into circular Theoremes, which Order he hath also observed in the Definitions and Suppositions of this Book. Moreover, he useth a certain way of instructing, which proceeds not by Composition, but by Resolution, as *Pappus* hath amply set down in his Commentaries on this Book.

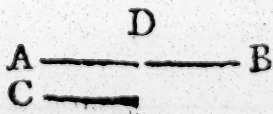
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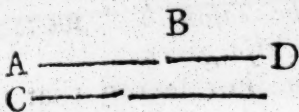
DEFINITIONS.

- I.  Lanes or Spaces, Lines, and Angles, to which we may find others equal, are said to be given by Magnitude.
- II. A Reason is said to be given, when we may find one of the same or equal thereto.
- III. Rectiline figures, whose angles are given, and also the reason of the sides to one another, are said to be given by Species or Kind.
- IV. Points, Lines and Angles, which have and keep always one and the same place and situation, are said to be given by Position or Situation.
- V. A Circle is said to be given by Magnitude, when the semidiameter thereof is given by Magnitude.
- VI. A Circle is said to be given by Position, and by Magnitude, when the center thereof is given by Position, and the semidiameter by Magnitude.
- VII. Segments of Circles, whose angles and bases are given by Magnitude, are said to be given by Magnitude.
- VIII. Segments of a Circle, whose angles are given by Magnitude, and the bases of the segments by Position and Magnitude, are said to be given by Position and by Magnitude.
- IX. A Magnitude AB , is greater than another Magnitude C , by a given Magnitude BD , when having taken away the given

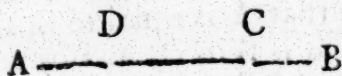


given Magnitude DB , the rest AD , is equal to the other Magnitude C .

- X. A Magnitude AB , is less than another Magnitude C , by a given Magnitude BD , when having added thereto the given Magnitude BD , the whole AD is equal to the other Magnitude C .



- XI. A Magnitude AB , is said to be greater than another magnitude CB , by a given magnitude AD , and in reason, when taking from the same magnitude the given magnitude AD , the rest DB , hath to the other magnitude CB , a given reason.



- XII. A magnitude AB is said to be less than another magnitude BC by a given magnitude AD , and in reason, when the given magnitude AD being added thereto, the whole DB hath to the other magnitude BC , a given reason.

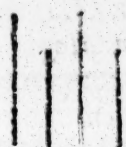


- XIII. A right line is said to be drawn down from a given point, unto a right line given in Position, the right line being drawn in a given angle.

- XIV. A right line is said to be drawn up from a given point, to a right line given in Position, the right line being drawn in a given angle.

- XV. A right line is against another right line in Position, when it is drawn parallel thereto through a given point.

PROPOSITION I.



TWO magnitudes *A* and *B* being given, the reason they have to one another *A* to *B* is also given.

a 1. def.

A B C D

magnitude *A* is given, a we may find one equal thereto, which let be *C*.

Again, forasmuch as the magnitude *B* is given, we may also find one equal to that, and let that be *D*. Therefore seeing that *A* is equal to *C*, and *B* to *D*, as *A* is to *C*, *b* so is *B* to *D*, and by permutation, *c* as *A* shall be to *B*, so *C* shall be to *D*.

b 7. 5.

c 16. 5.

d 2. def.

D. Therefore *d* the reason of *A* to *B* is given, for it is the same reason as of *C* to *D*, as we have found, and which ought to be demonstrated.

PROP. II.

If a given magnitude *A*, hath to some other magnitude *B*, a given reason, that other magnitude *B*, is also given by magnitude.



a 2. def.

A B C D

Demonstr. For seeing that *A* is given, we may find one equal thereto, which let be *C*: And forasmuch as the reason of *A* to *B*, is also given, we may find *a* one of the same. Let it be found, and let the

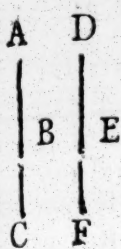
reason be of *C* to *D*. Now seeing that as *A* is to *B*, so *C* is to *D*; and by permutation, as *A* is to *C*, so *B* is to *D*: But *A* is equal to *C*, therefore *b* *B* shall be also equal to *D*. Therefore *c* the magnitude *B* is given, seeing that thereto there hath been found one equal, to wit, *D*.

b 14. 5.

c 1. def.

PROP.

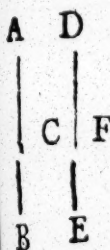
PROP. III.



If given magnitudes AB and BC , are compounded, that magnitude AC , that is compounded of them, shall be also given.

Demonstr. For seeing that AB is given, we may find one equal to it, which let be DE . Again, seeing that BC is given we may also find one equal to that, which let be EF . Wherefore seeing that DE is equal to AB , and EF is equal to BC , the whole AC *a 2. ax. 1.* is equal to the whole DF . Therefore AC is given, seeing that DF is proposed equal thereto.

PROP. IV.



If from a given magnitude AB , there be taken away a given magnitude AC , the remaining magnitude CB is also given.

Demonstr. Forasmuch as AB is given, we may find one equal thereto, which let be DE . Again, seeing that AC is given, we may also find one equal to it, which let be DF . Seeing then that the magnitude AB is equal to the magnitude DE , and the magnitude AC to the magnitude DF ; the rest CB *a 3. ax. 1.* shall be equal to the rest FE . Wherefore CB is given, for to it there hath been found an equal, to wit, FE .

PROP. V.



If a magnitude AB , hath a given reason to some part thereof AC , it will have also a given reason to the part remaining CB .

Demonstr. Let DE be exposed as a given magnitude, and seeing that the reason of the magnitude AB ,

- a 2. *def.* to the magnitude AC, is given, *a* we may find one of the same, which let be DE to DF; therefore the reason of the same DE to DF is given.
- b 2. *prop.* But DE being given, so is *b* also its part DF;
- c 4. *prop.* and consequently, *c* the rest FE: Therefore *d* seeing that DE and FE are given, the reason of the same DE to FE, is also given. And forasmuch as DE is to DF, as AB is to AC, and by conversion, as DE to FE, so AB is to CB. But the reason of DE to FE is given, as hath been demonstrated; therefore the reason of AB to CB is also given.
19. 5.

Scholium.

From this it is evident that if a magnitude hath to some part thereof a given reason, by division, the reason that one part hath to the other, shall be also given. For seeing that as DE is to FE, so is AB to CB; by division, as DF to FE, so AC to CB. But it hath been demonstrated that the parts DF and EF are given, and consequently their reason is also given: In like manner, therefore, the reason of AC to CB is given.

P R O P. VI.



If two magnitudes AB and BC, having to one another a given reason, are compounded, the magnitude AC compounded of them, shall also have a given reason to each of them AB and BC.

Demonstr. Let the given magnitude DE be exposed, and seeing that the reason of AB to BC is given; let

- there be made one and the same of the said DE to EF; therefore the reason of the same DE to EF is given; and therefore *a* the magnitude DE being given, both the one and the other of them
- a 2. *prop.* DE and FE, is given. Wherefore *b* the whole
- b 3. *prop.* DE and FE, is given. Wherefore *b* the whole
- c 1. *prop.* DF shall be also given. Therefore *c* the reason

of the same DF to each of them DE and EF, shall be given. And forasmuch then as AB is to BC, so is DE to EF; in compounding, *d* as AC is to BC, so is DF to EF: Therefore by conversion, as AC to AB, so is DF to DE. Therefore as the whole DF is to each of the other magnitudes DE and EF, so the whole AC is to each of the magnitudes AB and BC. Therefore *e* the reason of the same AC to each of the magnitudes AB and BC is given. *d* 18. 5. *e* 2. def.

PROP. VII.

C
A ——— B *If a given magnitude AB be divided according to a given reason AC to CB, each segment AC and CB is given.*

Demonstr. For seeing the reason of AC to CB is given, the reason of *a* AB to each of them *a* 6. prop. (AC and CB) is also given. But AB is given: Therefore *b* each of the segments AC and CB is *b* 2. prop. also given.

PROP. VIII.

A	D	<i>Magnitudes A and C, which have to one and the same a given reason B, shall be to one another in a given reason A to C.</i>
B	F	
C	E	
—	—	

Demonstr. For let the given magnitude D be exposed, and seeing that the reason of A to B is given, let the same be done of the said D to E. Now seeing that D is given, *a* E is also given. *a* 2. prop. Again seeing that the reason of B to C is given, let the same be done of E to F. But E is given, and therefore F is also given. But seeing that D is given, *b* the reason of the same D to F is given. *b* 1. prop. *ven*;

c 22. 5.

ven; and seeing that as A to B, so D to E, and as B to C, so is E to F; in reason of equality, c as A is to C, so is D to F; but the reason of D to F is given. Therefore the reason of A to C is also given.

PROP. IX.

<u>A</u>	<u>D</u>
<u>B</u>	<u>E</u>
<u>C</u>	<u>F</u>

If two or more magnitudes A, B, and C, are to one another in a given reason, and that the same magnitudes A, B, and C, have to other magnitudes D, E, and F, given reasons, altho they be not the same, those other magnitudes D, E, and F shall be also to one another in given reasons.

Demonstr. Forasmuch as the reason of A to B is given, as also that of A to D, the reason of D to B shall be given: But the reason of B to E is also given; therefore the reason of the same D to E shall be in like manner given. Again, seeing that the reason of B to C is given, and also that of B to E, the reason of E to C shall be given. But the reason of C to F is also given. Therefore a the reason of E to F shall be given. But it hath been demonstrated that the reason of D to E is also given; and therefore b the reason of D to F shall be given. Therefore the magnitudes D, E, and F, are to one another in given reasons.

PROP. X.

A ——— D B ——— C

If a magnitude AB, be greater than another magnitude BC, by a given magnitude, and in reason, the magnitude AC compounded of both, shall be also greater than

than that same magnitude, by a given magnitude, and in reason : But if that compounded magnitude be greater than the same magnitude, by a given magnitude, and in reason ; either the remainder shall be also greater than that same by a given magnitude and in reason ; or else the same remainder is given with the following, to which the other magnitude hath a given reason.

Demonstr. For seeing that AB is greater than BC by a given magnitude, and in reason, let the given magnitude AD be taken away. Therefore *a* the reason of the remainder DB to BC is given ; and in compounding, *b* the reason of DC to BC is also given. But the magnitude AD as also given ; therefore AC is greater than the same BC by a given magnitude, and in reason.

a 11. def.
b 6. prop.

Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in reason :

I say, that the
rest AB, is either greater

than the same BC by a given magnitude, and in reason ; or that the same AB, with that which followeth, to which BC hath a given reason, is given.

Forasmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in reason, cut off from it the given magnitude : Now the same given magnitude is either less than the magnitude AB, or greater : Let it in the first place be less, and let it be AD. Therefore the reason of the remainder DC to CB is given. Wherefore by division, the reason of DB to BC is given. But the magnitude AD is also given ; therefore the magnitude AB is greater *c* than the magnitude BC by a given magnitude, and in reason. Now let the given magnitude be greater than the magnitude

c 11. def.

A a

tude

tude AB, and let AE be put equal thereto; therefore *d* the reason of the remainder EC to CB is given; and by conversion, *e* the reason of the same BC to BE, is also given. But the same EB with BA is given, for that the whole AE is given: Therefore there is given AB, with that which follows BE, to which BC hath a given reason.

P R O P. XI.

E D B

A — 1 — 1 — 1 — C

If a magnitude AB, be greater than a magnitude BC, by a given magnitude, and in reason, the same magnitude AB, shall be also greater than the magnitude compounded of them by a given magnitude, and in reason, and if the same magnitude be greater than the two others together by a given magnitude, and in reason, that same magnitude shall be also greater than the rest by a given magnitude, and in reason.

Demonstr. For seeing that the magnitude AB is greater than BC by a given magnitude and in reason; let there be taken from it a given magnitude AD: Therefore *a* the reason of the rest DB to DC, is given, and therefore *b* the reason of DC to BD shall be also given: Let the same be done of AD to DE, therefore the reason of the same AD to DE is given. But AD is given, therefore *c* DE is also given, and consequently, *d* the rest AE, is also given. But seeing that as AD is to DE, so is DC to ED; by permutation, *e* as AD is to DC, so is DE to DB: Therefore by compounding, *f* as AC is to CD, so is EB to DB; and by permutation, *g* as AC is to EB, so is DC to DB. But the reason of DC to DB is given: Therefore also is AC to EB, and consequently, that of EB to AC. But it hath been demonstrated that AE is given, therefore *h* AB is greater than AC by a given magnitude, and in reason.

But

But now let AB be greater than AC by a given magnitude, and in reason: I say, that the same AB is also greater than the rest BC by a given magnitude, and in reason.

For seeing that AB is greater than AC by a given magnitude, and in reason. Let the given magnitude AB be cut off there-from: Therefore the reason of the remainder EB to AC is given, and consequently also shall be given that of AC to EB. Let the same be done of AD to DE, therefore the reason of AD to DE is given; and by conversion, the reason of AD to AE shall be also given, and consequently that of AE to AD. Now AE is given, therefore the whole AD shall be also given; and seeing that as the whole AC is to the whole EB, so the part cut off AD, is to the part cut off ED, so also shall be the remainder DC to the remainder DB. But the reason of AC to EB is given: Therefore also shall be given that of DC to DB. Wherefore by division, the reason of BC to DB is given; and consequently also shall be given that of DB to BC. But it hath been demonstrated that AD is given: Therefore AB is greater than the same BC by a given magnitude, and in reason.

i 11. def.

k 5. prop.

l 2. prop.

m 19. 5.

n schol. 5. prop.

o 11. def.

PROP XII.



If there be three magnitudes AB, BC, and CD, and that the first

AB, with the second BC, to wit, AC, be given. But the second BC, with the third CD, to wit, BD, be also given: Either the first AB shall be equal to the third CD, or the one shall be greater than the other by a given magnitude.

Demonstr. Forasmuch as each of the magnitudes AC and BD are given, the given magni-

A 2 2

tudes

- tudes are either equal to one another, or unequal. Let them be first equal: Therefore AC is equal to BD; take away the common magnitude BC, and there will remain *a* AB, equal to CD. But suppose them to be unequal as in this second figure, and let BD be greater than AC: Let then BE be put equal to AC. Now seeing that AC is given, BE is also given. But the whole BD is also given, the rest ED *b* shall be so also; and forasmuch as BE is equal to AC, taking away the common magnitude *c* BC, there will remain AB equal to CE. But ED is given: Therefore CD is greater than AB by the given magnitude ED.
- a 3. ax. 1.
- b 4. prop.
- c 3. ax. 1.

Scholium.

And in the first with the second, to wit, AC, were greater than the second with the third, to wit, BD, as in the other figure, CE would be made equal to the same BD, and by the same reasons as were above demonstrated, that AE is given and equal to CD; and therefore AB greater than CD by a given magnitude.

P R O P. XIII.

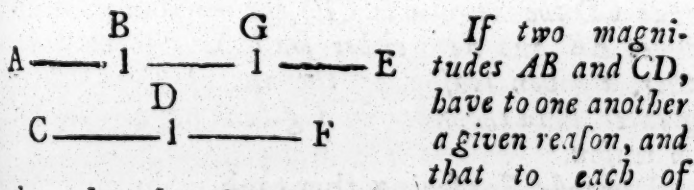
If there be three magnitudes AB, CD, and E, and that the first of them AB, bath a given reason to the second CD; but the second CD is greater than the third E, by a given magnitude, and in reason; also the first AB, shall be greater than

A ——— H ——— B
 1 ———
C ——— F ——— D
 1 ———
 E ———

than the third *E*, by a given magnitude, and in reason.

Demonstr. For seeing that *CD* is greater than *E* by a given magnitude, and in reason; let the given magnitude *CF* be taken therefrom: Therefore the reason of the rest *FD* to *E* is given. And forasmuch as the reason of *AB* to *CD* is given, let the same be done of *AH* to *CF*. Therefore the reason of the same *AH* to *CF* is given. But *CF* is given: Therefore *a* *AH* is also given. And seeing that as the whole *AB* is to the whole *CD*, so the part cut off *AH* is to the part cut off *CF*, and so also the rest *HB* is to the rest *FD*, the reason of the same *HB* to *FD* is also given. But the reason of *FD* to *E* is also given: Therefore *c* the reason of *HB* to *F* is given. But it hath been demonstrated that *AH* is given: Therefore *d* *AB* is greater than the said *E* by a given magnitude, and in reason. *a* 2. prop.
b 19. 5.
c 8. prop.
d 11. def.

P R O P. XIV.



If two magnitudes *AB* and *CD*, have to one another a given reason, and that to each of

them there be added a given magnitude, to wit, *BE* and *DF*; either the whole *AE* and *CF* shall have to one another a given reason, or the one shall be greater than the other by a given magnitude, and in reason.

Demonstr. For seeing that each of those magnitudes *BE* and *DF*, is given, *a* the reason of the said *BE* and *DF* is also given; and if that reason be the same with that of *AB* to *CD*, that of the whole *AE* to the whole *CF*, *b* shall be the same; and therefore the reason of the said *AE* to *CF* is given. *a* 1. prop.
b 12. 5.

A a 3

Now

- Now let the reason of BE to DF be not the same with that of AB to CD, and let it be as AB to CD, so BG to DF. Therefore the reason of the said BG to DF is given. But
 e 2. prop. the magnitude DF is given, therefore c BG is also given; and seeing that the whole BE
 d 4. prop. is given, d the rest GE shall be also given. But forasmuch as AB is to CD, as BG is to DF, e so also is the whole AG to the whole CF; and therefore the reason of the said AG to CF is given: But the magnitude GE is
 e 12. 5. given: Therefore f the magnitude AE is greater than the magnitude CF by a given magnitude, and in reason.

P R O P. XV.

$$\begin{array}{c} \text{E} \quad \text{G} \\ \text{A} \text{ --- } | \text{ --- } | \text{ --- } \text{B} \\ \text{F} \\ \text{C} \text{ --- } | \text{ --- } \text{D} \end{array}$$

If two magnitudes AB and CD, have to one another a given reason, and that from each of them be taken away a given magnitude (to wit. from the magnitude AB the magnitude AE, and from the magnitude CD the magnitude CF) the remaining magnitudes EB and FD, either shall have to one another, a given reason, or the one of them shall be greater than the other by a given magnitude, and in reason.

- Demonstr.* For seeing that each magnitude AE and CF is given, the reason of AE to CF is given; and if it be the same with that of AB to CD, that of the remainder EB to the remainder FD, a shall be also the same; and therefore the reason of the said EB to FD shall be also given.

$$\begin{array}{c} \text{E} \quad \text{G} \\ \text{A} \text{ --- } | \text{ --- } | \text{ --- } \text{B} \\ \text{F} \\ \text{C} \text{ --- } | \text{ --- } \text{D} \end{array}$$

But if it be not the same, let it be as AB to CD, so AC to CF. Now the reason of AB to CD is given, therefore

therefore also that of AG to CF shall be given.
 But CF is given, therefore *b* AG is given. *b* 2. prop.
 But AE is also given, therefore *c* the rest EG *c* 4. prop.
 is given; and seeing that as AB is to CD, so
 the part cut off AG is to the part cut off CF,
 and so also is *d* the rest GB to the rest FD; the *d* 4. prop.
 reason of the said GB to FD is also given.
 Therefore seeing that EG is given, EB is
 greater than FD *e* by a given magnitude, and *e* 11. def.
 in reason.

P R O P. XVI.

$$\begin{array}{c} \text{G} \quad \text{B} \\ \text{A} \text{---} \text{I} \text{---} \text{I} \text{---} \text{F} \\ \text{E} \\ \text{C} \text{---} \text{---} \text{I} \text{---} \text{D} \end{array}$$

If two magnitudes AB and CD, have to one another a given reason, and that from one of them, to wit, CD, there be taken away a given magnitude DE, and to the other AB there be added a given magnitude BF, the whole AF shall be greater than the rest CE, by a given magnitude, and in reason.

Demonstr. For seeing that the reason of AB to CD is given. let the same be made of BG to DE: Therefore *a* the reason of the said BG *a* 2. def.
 to DE is given. But DE is given, therefore
b BG is also given. But BF is also given, *b* 2. prop.
 therefore *c* the whole GF is given. And see- *c* 3. prop.
 ing that as AB is to CD, so the part cut off
 BG, is to the part cut off DE; and *d* so also *d* 19. 5.
 is the remainder AG to the remainder CE;
 the reason of the said AG to CE is given:
 But GF is given, therefore the magnitude AF
 is greater than the magnitude CE by a given
 magnitude, and in reason,

P R O P. XVII.

$$\begin{array}{c}
 \text{F} \\
 \text{A} \text{ --- } | \text{ --- } \text{B} \\
 \text{--- } | \text{ --- } \\
 \text{E} \\
 \text{C} \text{ --- } | \text{ --- } \text{D} \\
 \text{--- } | \text{ --- } \\
 \text{G}
 \end{array}$$
 If there be three magnitudes AB, E, and CD, and that the first AB be greater than the second E by a given magnitude, and in reason: But the third CD be also greater than the same second E, by a given magnitude, and in reason; the first AB shall have to the third CD, either a given reason, or else the one shall be greater than the other by a given magnitude, and in reason.

Demonstr. For seeing that AB is greater than E by a given magnitude, and in reason, let the magnitude AF be taken away: Therefore the reason of the remainder FB to E is given. Again, seeing that CD is greater than the said E by a given magnitude, and in reason, let the given magnitude CG be cut off therefrom; and the reason of the remainder GD to E shall be given: Therefore a the reason of FB to GD shall be also given. But to the said FB and GD are added the given magnitudes AF and CG: Therefore the whole AB and CD shall either have to one another a given reason, or the one shall be greater than the other by a given magnitude, and in reason.

a 8. prop.

b 14. prop.

P R O P. XVIII

$$\begin{array}{c}
 \text{B} \\
 \text{A} \text{ --- } | \text{ --- } \text{H} \\
 \text{I} \quad \text{G} \\
 \text{C} \text{ --- } | \text{ --- } | \text{ --- } \text{D} \\
 \text{F} \\
 \text{E} \text{ --- } | \text{ --- } \text{K}
 \end{array}$$
 If there be three magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or EF, by a given magnitude, and in reason; either the two others AB and EF, shall have

have to one another a given reason, or the one shall be greater than the other by a given magnitude, and in reason

Demonstr. Forasmuch as the magnitude CD is greater than the magnitude AB by a given magnitude, and in reason, let the given magnitude DG be taken therefrom: Therefore the reason of the remainder CG to AB is given. Let the same be made of GD to BH, therefore the reason of the said DG to BH is given. But DG is given, therefore *a* EH is also given. *a 2. prop.* And seeing that as CG is to AB, so is GD to BH, *b* so also is the whole CD to the whole AH, the reason of the said CD to AH shall be also given. *b 12. 5.*

Again, seeing that the same CD is greater than EF by a given magnitude, and in reason; let the magnitude DI be cut off therefrom: Therefore the reason of the remainder CI to EF is given: Let the same be made of DI to FK. Therefore the reason of the said DI to FK shall be also given. But DI is given, therefore FK is also given. And seeing that as CI is to EF, so is ID to FK; so also is the whole *c* CD to the whole EK; the reason of *c 12. 5.* the said CD to EK shall be given. But the reason of the same CD to AH is also given: Therefore *d* the reason of the said AH to EK *d 3. prop.* shall be given. And seeing that from the said AH and EK, the given magnitudes BH and FK are cut off, the magnitudes AB and EF *e* are either in a given reason to one another, or the one is greater than the other by a given magnitude, and in reason. *e 15. prop.*

P R O P.

PROP. XIX.



If there be three magnitudes AB , CD , and E , and that the first AB , be greater than the second CD , by a given magnitude; and in reason; and that the second CD be

greater than the third E , by a given magnitude, and in reason; also the first magnitude AB shall be greater than the third E , by a given magnitude, and in reason.

Demonstr. For seeing that CD is greater than E by a given magnitude, and in reason; let the given magnitude, CF be taken therefrom: Therefore the reason of the remainder FD to E is given. Again, seeing that AB is greater than the same CD by a given magnitude, and in reason: Let the magnitude AG be taken therefrom: Therefore the reason of the remainder GB to CD is given: Let the same be made of GH to CF : Therefore the reason of the said GH to CF is given. But CF is given: Therefore also GH is given, and then AG is also given, the whole AH shall be also given.

a 3. prop.



b 19. 5.

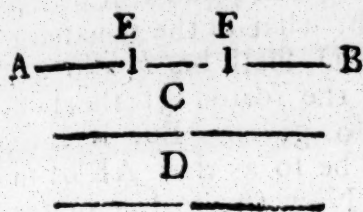
But as GB is to CD , so is GH to CF , and so also b the remainder HB to the remainder FD : Therefore the reason of the said HB to

FD is given. But the reason of the same FD to E is also given: Therefore the reason of HB to E is in like manner given, and so is also the magnitude AE : Wherefore the magnitude AB *c* is greater than E by a given magnitude, and in reason.

c 11. def.

OTHER.

OTHERWISE.

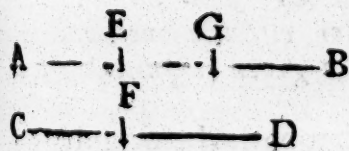


Construction. Let there be three magnitudes AB, C, and D, and let AB be greater than C by a given magnitude, and in reason; but

let C be also greater than D, by a given magnitude, and in reason: I say, that AB is greater than D by a given magnitude, and in reason.

Demonstr. Forasmuch as AB is greater than C by a given magnitude, and in reason, let the given magnitude AE be cut off therefrom: Therefore the reason of the remainder EB to C is given. But the magnitude C is greater than the magnitude D by a given magnitude and in reason; therefore EB is greater than D by a given magnitude, and in reason: Wherefore let the given magnitude EF be cut off therefrom; and the reason of the remainder FB to D shall be given. But AF is given. Therefore AB is greater than D by a given magnitude, and in reason. d 13. prop. e 3. prop.

PROP. XX.



If there be two given magnitudes, AB and CD, and that from them there be taken magnitudes AE

and CF, having to one another a given reason; either the remaining magnitudes EB and FD, shall have to one another given reasons; or else the one shall be greater than the other by a given magnitude, and in reason.

Demonstr.

- Demonstr.* For seeing that both the magnitudes AB and CD, are given, the reason of the said AB to CD is *a* also given; and if it be the same as of AE to CF, that of the remainder EB to the remainder FD shall be *b* also the same; and therefore the reason of the said EB to FD shall be also given. But if it be not the same, let it be so as that AE be to CF, as AG to CD. Now the reason of the said AE to CF is given: Therefore the reason of the said AG to CD is given. But CD is given, therefore *c* AG is also given. But the whole AB is likewise given, therefore *d* the remainder BG is given. And seeing that as AE is to CF, so is AG to CD, and also the remainder EG to the remainder FD, the reason of the said EG to FD is given. But GB is also given: Therefore the magnitude EB is greater *e* than the magnitude BD by a given magnitude, and in reason.
- a* 1. *prop.*
b 19. 5.
c 3. *prop.*
d 4. *prop.*
e 11. *def.*

P R O P. XXI.

$$\begin{array}{ccccccc} & G & & B & & & \\ A & \text{---} & I & \text{---} & I & \text{---} & E \\ & & & & D & & \\ C & \text{---} & \text{---} & \text{---} & I & \text{---} & F \end{array}$$

If there be two magnitudes given AB and CD, and to them be added other magnitudes BE and DF, having to one another a given reason; either the whole AE and CF shall have to one another a given reason, or else the one shall be greater than the other by a given magnitude, and in reason.

- Demonstr.* For seeing that both the magnitudes AB and CD are given, their reason *a* is also given; and if it be the same reason as of BE to DF, the reason of the whole AE to the whole CF shall be also given; for it shall be *b* the same. But if it be not the same, let it be as BE is to DF, so BG to CD: Therefore the reason of the said BG to CD is given. But
- a* 1. *prop.*
b 12. 5.

But CD is given; therefore *c* also BG shall be given. But the whole AB is given; therefore also the *d* remainder AG shall be given. And seeing that as BE is to DF, so is BG to CD, and also *e* the whole GE to the whole CF, the reason of the said GE to CF shall be likewise given. But AG is given; therefore the magnitude AE is greater than the magnitude CF by a given magnitude, and in reason. *c* 2. *prop.* *d* 4. *prop.* *e* 12. 5.

P R O P. XXII.

$\begin{array}{c} B \\ A \text{ --- } | \text{ --- } C \\ D \text{ ---} \end{array}$ If two magnitudes AB and BC, have to some other magnitude D, a given reason, also their compound magnitude AC, shall have to the same magnitude D, a given reason.

Demonstr. For seeing that each magnitude AB and BC hath a given reason to D, the reason *a* of AB to BC is given; and by compounding, *a* 8. *prop.* *b* the reason of AC to BC is given. But that *b* 6. *prop.* of BC to D is also given, therefore *c* the reason of the said AC to D shall be likewise given. *c* 8. *prop.*

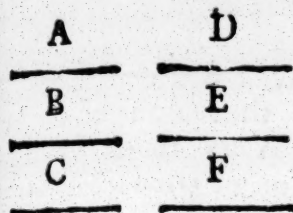
P R O P. XXIII.

$\begin{array}{c} E \\ A \text{ --- } | \text{ --- } B \\ F \quad G \\ C \text{ --- } | \text{ --- } D \end{array}$ If the whole AB be to the whole CD in a given reason, and that the parts AE and EB be to the parts CF and FD in given reasons, altho' they be not the same, the whole (to wit, AB, AE, and BE,) shall be to the whole (to wit, CD, CF, and FD,) in given reasons.

Demonstr. For seeing that AE is to CF in a given reason, let the same be made of AB to CG; therefore the reason of the said AB

219. 5. to CG is given; and consequently also that *a* of the rest EB to the rest FG. But the reason of FD to the same EB is also given: Therefore the reason of FD to FG *b* is likewise given; and therefore *c* that of FD to the remainder GD is also given. But the reason of AB to each of the magnitudes CD and CG is given: Therefore *d* also the reason of CD to CG is given, and again *e* that of CD to the remainder GD. But the reason of FD to DG is given, therefore also *f* that of the same CD to FD, and consequently that of *g* CD to the remainder FC; and therefore also the reason of CF to FD shall be given. But the reason of EB to FD is proposed to be given; therefore the reason of CF to EB shall be given. Again, for that the reason of AB to CD is given; and also that of CD to each of those FC and FD, the reason of the same AB to each of the said FC and *h* FD, shall be likewise given. But the reason of the said FD to EB is given: Therefore the reason of AB to BE shall be also given, and consequently AB to the remainder *i* AE. Wherefore by division *k* the reason of AE to EB shall be likewise given. But the reason of EB to FD is given. Therefore also that of AE to FD. In like manner, seeing that the reason of CD to AB is given; and that of AB to each of his parts AE and EB; also the reason of the said CD to each of the said AE and EB, *l* shall be given: Wherefore each of the magnitudes AB, CD, AE, EB, CF, and FD, is to each of the others in a given reason.

PROP. XXIV.



If of three right lines A, B, and C, proportional A to B, as B to C, the first A hath to the third C a given reason, it will also have to the second B a given reason.

Demonstr. For, let there be exposed another right line D, and seeing that the reason of A to C is given; let the same be made of D to F; therefore the reason of D to F is given. But D is given, therefore F is also given; betwixt the two right lines D and F, let there be taken a mean proportional E. Therefore the rectangle made under D and F is equal to the square of E. But the same rectangle of D and F is given: (for all the angles of that rectangle are given, being right angles, and the reasons that the sides have to one another are also given;) therefore the square of E is given, and consequently the same right line E is also given (for one equal thereto may be found, seeing that the rectangle of D and F is given.) But D is given, therefore the reason of D to E is given, and as A is to C, so D is to E. But as A is to C, so the square of A is to the rectangle of A and C, and also as D is to E, so the square of D is to the rectangle of D and E. Therefore as the square of A is to the rectangle of A and C, so the square of D is to the rectangle of D and E. But the rectangle of A and C is equal to the square of B, (seeing that A, B, and C, are proportional) and that of D and E to the square of E, therefore as the square of A is to the square of B, so the square of D is to the square of E: Where.

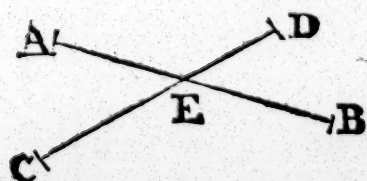
g 2. 6. Wherefore *g* as *A* is to *B*, so *D* is to *E*. But
 h 2. def. the reason of *D* to *E* is given, therefore *h* also
 the reason of *A* to *B* is given.

O T H E R W I S E.

Demonstr. Forasmuch as the reason of *A* to
C is given, and that as *A* is to *C*, so the
 square of *A* and *C*, the reason of the said square
 of *A* to the rectangle made of *A* and *C*, is also
 given. But to the rectangle made of *A* and *C*

$\begin{array}{c} A \\ \hline B \\ \hline C \\ \hline \end{array}$	the square of <i>B</i> is equal (seeing that <i>A</i>, <i>B</i> and <i>C</i>, are proportional;) therefore the reason of the square of <i>A</i> to the square of <i>B</i> is given; and by consequence, the reason of the line <i>A</i> to the line <i>B</i> is given; for to each of them <i>A</i> and <i>B</i>, we have exhibited an equal to the pro- per square of each one.
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P R O P. XXV.



If two lines *AB* and
CD, given by position
do intersect the point
E in which they inter-
sect one another, is gi-
ven by position.

Demonstr. For if it change its place, the one
or the other of the lines *AB* and *CD*, would
change its position: But so it is that by Sup-
position it changeth not: Therefore *a* the point
E is given by position.

a 4. def.

P R O P. XXVI.

A ——— *B* If the extremities *A* and *B*, of
a right line *AB*, be given by posi-
tion, that same right line *AB* is given by position
and by magnitude.

Demonstr.

Demonstr. For if the point A remaining in its place, the position, or the magnitude of the right line AB shall change, the point B will fall elsewhere. But so it is, that by Supposition it doth not fall elsewhere. Therefore the right line AB is given by position, and by magnitude.

P R O P. XXVII.

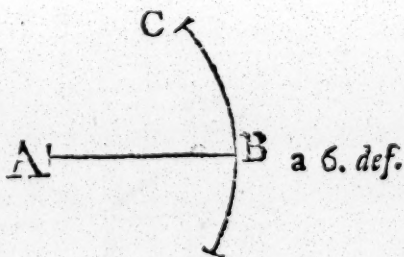
If one of the extremes A of A ——— B a right line AB, given by position and magnitude, be given, the other extremity B shall be also given.

Demonstr. For if the point A remaining in its place, the point B shall change and fall in some other place, either the position of the right line AB, or its magnitude would change: But so it is that according to the Supposition, neither the one nor the other doth change. Therefore the point B is given.

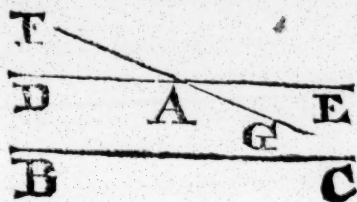
O T H E R W I S E.

Constr. On the center A, with the distance AB, describe the circumference BC.

Demonstr. Therefore a that circumference BC is given by position. But the right line AB is also given by position; therefore the point b B is b 25. prop. given.



PROP. XXVIII.

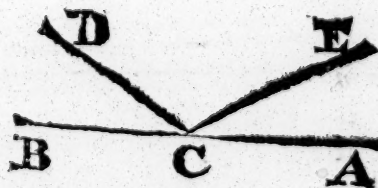


If by the given point *A*, there be drawn a right line *DAE*, against another right line *BC*, given by position, the right line *DAE* so drawn, is given by position.

a 13. def.
b 30. 1.

Demonstr. For if it be not given, the point *A* remaining in its place, the position of the right line *DAE* may change: Let it then change if it be possible, and fall elsewhere, remaining parallel to *BC*, and let it be the line *FAG*: Therefore *BC* is parallel to the said line *FAG*. But a the same *BC* is also parallel to *DAE*: Therefore b *DAE* is parallel to the said line *FAG*, which is absurd; seeing they join together, and meet in *A*: Therefore the position of the right line *DAE* falls not elsewhere. Wherefore the said line *DAE* is given by position.

PROP. XXIX.



If to a right line *AB*, given by position, and to a point *C* given therein, there be drawn a right line *CD*, which shall make

a given angle *ACD*, the line drawn *CD* is given by position.

Demonstr. For if it be not given by position, the point *C* remaining in its place, the position of the line *CD* observing the magnitude of the angle *ACD*, will fall elsewhere. Let it fall elsewhere then if it be possible, and let it

it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the lesser, which is absurd. Therefore the position of the right line CD, shall not fall elsewhere; and therefore the said line CD is given by position.

P R O P. XXX.

If from a given point A, be drawn to a right line BC, given by position, a right line AD, making a given angle ADB, that line drawn AD is given by position.



Demonstr. For if it be not given, the point A remaining in its place, the position of the right line AD keeping, magnitude of the angle ADB, will change. Let it change then, and let it be the right line AE: Therefore the angle ADB is equal to the angle AEB, the greater to the lesser, which is absurd. Therefore the position of the right line AD doth not change; and therefore the said line AD is given by position. a 16. 1.

O T H E R W I S E.

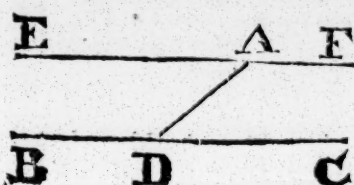
Constr. By the point A let there be drawn the line EAF, parallel to the right line BC.

Demonstr. Then seeing that by the given point A, and against the right line BC, given by position, there is drawn the right line EF, those lines EF and BC are parallels. But on

B b 2

the

b 29. r.



the same lines doth
also fall the right
line AD. Therefore
b the angle FAD is
equal to the given
angle ADB ; and

therefore it is also given. Wherefore to the
right line EF given by position, and to the
given point A therein, there is drawn the right
line AD, making the given angle FAD.

c 29. prop. Therefore c the said line AD is given by po-
sition.

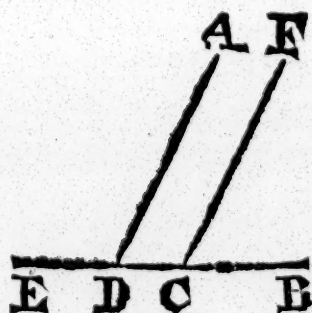
OTHERWISE.

Constr. In the line BCE, let there be taken
the given point C, and by the same let there
be drawn the line CF, parallel to the said DA.

Demonstr. Forasmuch as AD and CF are pa-
rallels, and that on them there doth fall the
right line BCE, the angle FCB is equal d to the
given angle ADB ; and therefore it is also given.
And seeing that the right line BC is given by

d 29. ::

e 29. prop.



f 28. prop. the line AD. Therefore the said line f AD is
given by position.

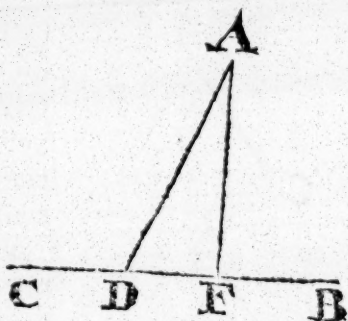
position, and that to a
given point C therein,
there is drawn the right
line FC, making the gi-
ven angle FCB, that
same line FC e is given
by position. But by the
given point A, opposite
to the line FC given by
position, there is drawn

OTHER.

OTHERWISE.

Constr. In the right line BC assume some point at F, and draw AF.

Demonstr. Forasmuch as each point A and F is given, the right line AF is given *g* by position. But the line BC is also given



g 26. *prop.*

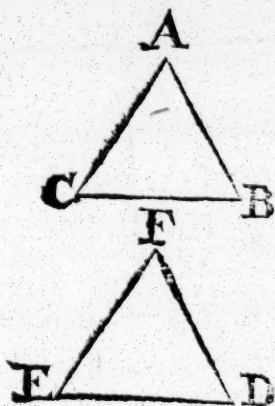
by position. Therefore * the angle AFD is given. But by supposition, the angle ADF is given: Therefore DAF (which is the residue *h* of two right angles) is given; and seeing *h* 32. 1. that to the right line AF given by position, and to the given point therein A there is drawn the right line DA, making the given *i* 29. *prop.* angle DAF, *i* that same line DA is given by position.

Scholium.

* EUCLIDE supposeth here that two right lines being given by position, and inclining to one another, do make a given angle, which some do demonstrate after this manner.

Demonstr. Forasmuch as the two right lines given by position, do incline to one another, the inclination of those lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right lines given by position, and inclining to one another, is given.

Another thus demonstrateth it.



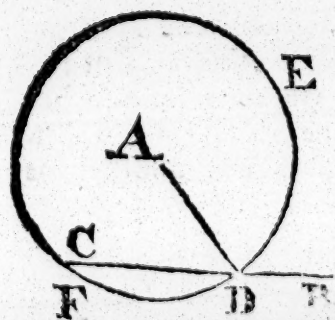
Constr. Let there be two right lines inclining to one another, as AB and CB, given by position; and in the line AB let there be taken a given point A, and in BC also some point, as C; and let the right line AC be drawn.

Demonstr. Seeing that as well the point B, as each of the points A and C is given, *k* 26. *prop.* the three right line AB, BC, and AC, are given by magnitude. Wherefore of three direct lines equal unto them, a triangle may be constituted: Let there then be made the triangle FDE, having the side FD equal to the side AB, the side FE equal to the side AC, and the base DE equal to the base BC.

Seeing then the angles comprised of equal right lines are equal, we have found the angle FDE equal to the angle ABC; and therefore the same angle ABC is given.

1 r. *def.*

P R O P. XXXI.



If from a given point A there be drawn to a right line given by position BC, a right line AD, given by magnitude, that line AD shall be also given by position.

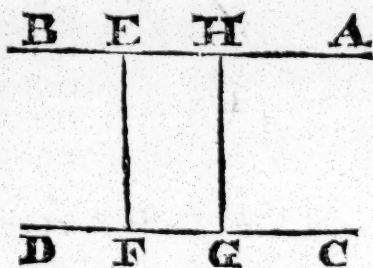
Constr. From the center A, with the distance AD, let the circle DEF be described.

Demonstr.

Demonstr. Forasmuch as the center A is given by position, and the semidiameter AD by magnitude, the circle DEF *a* is given by position. *a* 6. def.
But the right line BC is also given by position: Therefore the point of intersection D *b* is given, and seeing that the point A is also given: *b* 25. prop.
c the right line AD is given by position. *c* 26. prop.

P R O P. XXXII.

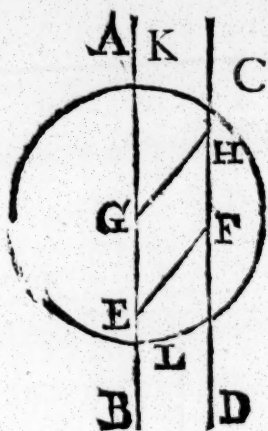
If unto parallel right lines AB and CD, given by position, there be drawn a right line EF, making the given angles BEF and EFD, the line drawn EF shall be given by magnitude.



Constr. For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

Demonstr. Forasmuch as the lines EF and HG are parallels, and that on them doth fall the line CD; *a* the angle EFD is equal to the angle FGH. But the angle EFD is given, therefore the angle FGH is also given. And forasmuch as to the right line CD given by position, and to the point G given in the same, there is drawn the right line GH, making the given angle FGH, *b* the said line GH is given by position. But AB is also given by position, therefore *c* the point H is given. But the point G is also given: Therefore *d* the line GH is given by magnitude, and is *e* equal to EF. Wherefore *f* the said line is given by magnitude. *a* 29. 1.
b 29. prop.
c 25. prop.
d 26. prop.
e 34. 1.
f 1. def.

PROP. XXXIII.



If unto parallel right lines AB and CD , given by position, there be drawn a right line EF given by magnitude, that line EF shall make the given angles BEF and DFE .

Constr. For let there be taken in the right line AB the point G , and by that point let there be drawn the line GH parallel to EF .

Demonstr. Therefore EF

a 34. 1.

is equal to the said a GH . But EF is given by magnitude, therefore GH is also given by magnitude. But the point G is given, and

a 6. def.

therefore if on that point, with the distance GH , there be described a circle, that circle shall be given by position: Let it be then described, and let it be HKL , the said circle HKL is therefore given by position. But the line CD which doth cut the circumference KHL in H , is also given by position. Therefore the said point of intersection H is given.

c 25 prop.

d 26. prop.

But the point G is given: Therefore the right line GH is given by position. But the right line CD is also given by position: Therefore the angle GHF is given. But to that

e schol. 30.

prop.

f 29. 1.

angle the angle EFD is equal: Therefore the angle EFD is given; and therefore also the angle BEF ; for that it is the residue of the sum of two right angles.

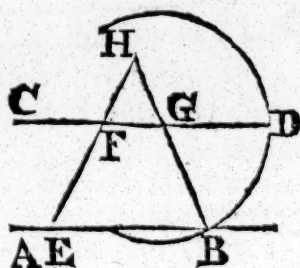
g 29. 1.

OTHERWISE.

Constr. Let there be taken in the right line CD the point G , and let GD be put equal to EF .

EF, then from the center G, with the distance GD, let there be described the circle HDB, and draw GB.

Demonstr. Forasmuch as the center G is given by position, and the semidiameter GD by magnitude, the circle BDH *h* is given by position. *h 6. def.* But the line AB is also given by position: Therefore *i* the point B is given. But the point G is also given, therefore *k* the right line GB is given by position. But the right line CD is also given by position: Therefore *l* the angle BGD is given. Wherefore if EF be parallel to BG, the angle EFD *m* shall be given, and consequently also the other angle BEF. *i 25. prop.*



k 26. prop.

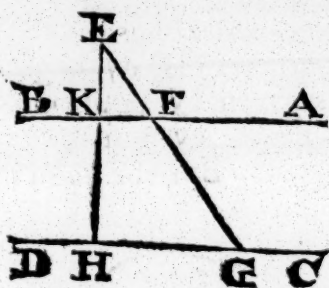
l sch. 30. prop.

m 29. r.

But the right lines BG and EF being not parallels, let them meet in the point H. Forasmuch as EB is parallel to FG, and EF is equal to GD, that is to say, to BG; also FH shall be equal to GH (for EH and BH being cut proportionally *o* by the parallel FG, as EF is to FH, so is BG to GH; and by permutation, as EF is to BG, so is FH to GH:.) *n 14. 5. o 2. 6.* Therefore *p* the angle HFG is equal to the angle HGF, but the said angle HGF is given (for that it is equal *q* to the given angle BGD:.) *p 5. r. q 15. r.* Therefore the angle HFG is also given. But to that angle the angle BEF is equal; and therefore is given, as also the remaining angle EFG.

PROP.

PROP. XXXIV.

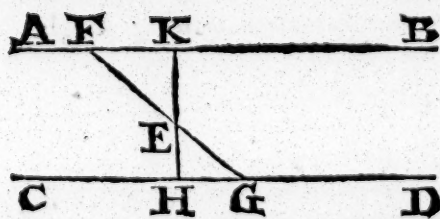


If from a given point *E*, there be drawn unto parallel right lines *AB* and *CD*, given by position, a right line *EFG*, that right line *EFG* shall be divided in a given reason (to wit) as *EF* to *FG*.

Constr. For from the point *E* let there be drawn the line *EH*, perpendicular to the line *CD*.

Demonstr. Forasmuch as from the given point *E* there is drawn to the line *CD* the right line *EH*, making the given angle *EHG* *a* the said line *EH* is given by position, but both the one and the other lines *AB* and *CD* is also given by position. Therefore *b* the points of intersection *K* and *H*, are given. But the point *E* is also given: Therefore *c* each line *EK* and *KH* is given. Wherefore *d* the reason of the said *EK* to *KH* is given. But as *EK* is to *KH*, so is *EF* to *FG*; (for in the triangle *GEH* the line *KF* being parallel to *HG*, the sides *EH* and *EG* are cut proportionally:) Therefore the reason of the said *EF* to *FG* is given.

OTHERWISE.



Constr. To the parallel right lines given by position, *AB* and *CD*, let there be drawn from the point *E* the right line *FEG*: I say, that the reason of *GE* to *EF* is given.

Demonstr.

Demonstr. For from the point E let there be drawn to CD the perpendicular EH, and produced to the point K; seeing therefore that from the point E to the right line CD, given by position, there is drawn the line EH, making the given angle EHG, *a* the said line EH is *a* 30. *prop.* given by position: But each line AB and CD is also given by position: Therefore *b* each *b* 25. *prop.* point of intersection H and K is given. But the point E is also given, therefore *c* each of *c* 26. *prop.* the lines EH and EK is given by magnitude; and therefore *d* the reason of the said EH to *d* 1. *prop.* EK is given. But *e* as EH is to EK, so is EG *e* 4. 6. to EF (for the opposite angles at the point E, being equal, and the lines AB and CD parallels, the triangles EHG and EKF are equiangled; and therefore as EH is to EG, so is EK to EF; and by permutation as EH to EK, so is EG to EF.) Therefore the reason of the said lines EG to EF is given.

P R O P. XXXV.

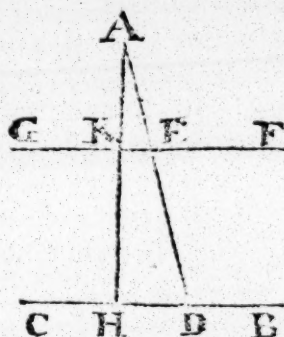
If from a given point A, to a right line BC, given by position, there be drawn a right line AD, which let be divided in E, in a given reason (to wit) as AE to ED, and that by the point of section E there be drawn a right line FEG, opposite to the right line BC, given by position, the line FG drawn shall be given by position.

Constr. For from the point A, let there be drawn the line AH, perpendicular to the line BC.

Demonstr. For seeing that from the given point A there is drawn to BC given by position, the right line AH making the given angle AHD, *a* the said line AH is given by position. *a* 30. *prop.* But BC is also given by position: Therefore *b* *b* 25. *prop.* the point H is given. But the point A is also given:

c 26. *prop.* given: Therefore *c* the line AH is given by magnitude and by position.

d 2. 6.



e 6. *prop.*

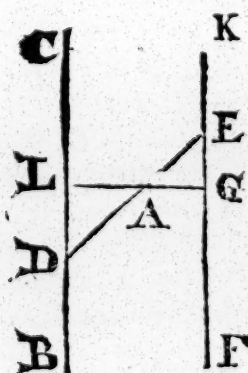
f 2. *prop.*

g 27. *prop.*

h 28. *prop.*

given by magnitude. But AK is also given by position, and the point A is given: Therefore *g* the point K is also given, and seeing that by the said given point K there is drawn the line FG, opposite to the right line BC given by position; the said line FG *h* is given by position.

P R O P. XXXVI.



If from a given point A, there be drawn to a right line BC given by position, a right line AD, and to it be added a right line AE, having to the same AD a given reason, and that by the extremity E of the added line AE, there be drawn a right line FEK, opposite to the line BC, given by position, that same line FEK shall be given by position.

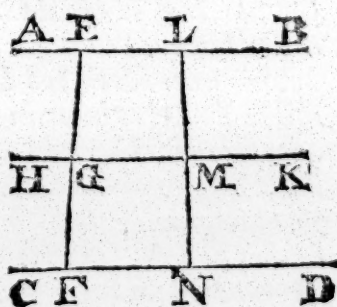
Constr. For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

Demonstr. Forasmuch as from the given point A, there is drawn to the right line BC, given

given by position, the right line GL, which makes the given angle GLD, *a* that line GL is *a* 30. *prop.* given by position. But BC is also given by *b* 26. *prop.* position, therefore *b* the point L is given; and seeing that the point A is also given the line *c* AL is given. But forasmuch as the rea- *c* 26. *prop.* son of AE to AD is given, and that *d* as the *d* 4.6. said AE is to AD, so is AG to AL; (because the triangles ALD and AGE are equiangled) the reason of AG to AL is also given. But AL is given by magnitude: Therefore *e* AG *e* 2. *prop.* is given by magnitude. But it is also given by position, and the point A is given: Therefore *f* the point G is also given. And seeing *f* 27. *prop.* that by the same given point G there is drawn the line FK, opposite to the right line BC, given by position, *g* the said line FK is given *g* 28. *prop.* by position.

P R O P. XXXVII.

If unto parallel right lines AB and CD, given by position there be drawn a right line EF, divided in the point G, in a given reason, (to wit, of EG to GF;) but if by the point of section G, there be drawn opposite to the



right lines AB or CD, given by position, a right line HGK, that line drawn shall be given by position.

Constr. For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

Demonstr. Seeing that from the given point L, there is drawn to the right line CD, the line LN, making the given angle LND, the laid

- a 30. *prop.* said LN *a* is given by position. But CD is also given by position: Therefore the point
 b 25. *prop.* N *b* is given. But the point L is also given:
 c 26. *prop.* Therefore *c* the line LN is given; and seeing that the reason of FG to GE is given, and that * as FG is to GE, so is NM to ML, the reason of the said NM to ML is given;
 d 6. *prop.* and in compounding, *d* the reason of LN to LM is also given. But LN is given by magnitude therefore ML is *e* given by magnitude.
 e 2. *prop.* But it is also given by position, and the point
 f 27. *prop.* L is given: Therefore the point M *f* is also given. And considering that by the said point M there is drawn the right line KH, opposite to the right line CD, given by position, the said line KH is also given by position.

Scholium.

* EUCLIDE supposeth here, that as FG is to GE, so NM is to ML; but by another it is thus demonstrated.

The lines EF and LN are parallels or not parallels: Let them in the first place be parallels, and forasmuch as by Construction the lines EL, FN, EF, and LN, are parallels, EN shall be a parallelogram; and therefore the side EF is equal to the side LN. Again, seeing that MG is parallel to NF, and GF to MN, GN shall be also a parallelogram; and therefore the side GF is equal to the side MN. Wherefore the equal sides EF and LN, shall have to the equal sides FG and MN, *g* one and the same reason. Therefore as EF is to FG, so is LN to MN; and in dividing, *h* as GE to GF, so is LM to MN.

g 7. 5.

h 17. 5.

Now suppose that the lines EF and LN be not

parallels, but that they meet in the point O . For-

asmuch as in the

triangle OFN there

is drawn HK , pa-

rallel to FN one of

the sides; i the sides

OF and ON are di-

vided proportionably;

and therefore as FG

is to GO , so is NM

to MO . Again, see-

ing that in the tri-

angle OGM there is

drawn EL , parallel to the side GM , the sides

OG and OM are divided proportionally: Wherefore

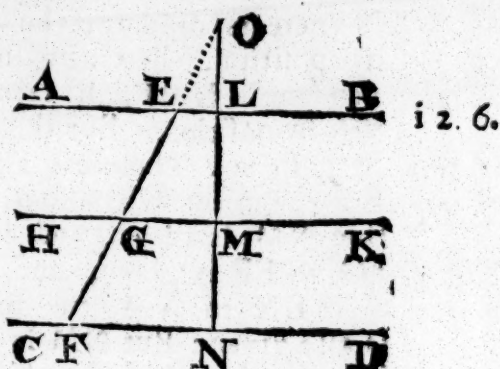
as OE is to EG , so is OL to LM ; and by com-

pounding, 1 as OG is to EG , so is OM to LM ; but

it hath been demonstrated that as FG is to GO , so

is NM to MO ; therefore in reason of equality, m

as FG is to GE , so is NM to ML .



P R O P. XXXVIII.



If unto parallel right lines AB and CD , there be drawn a right line EF , and that to it there be added some other right line EG , which hath a given reason to the same EF ; but if by the extremity G of the

added line EG , there be drawn a right line HK , against the parallels given by position AB and CD , the line drawn HK shall be also given by position.

Constr. For let there be taken in the line AB , the given point N , and from thence let there

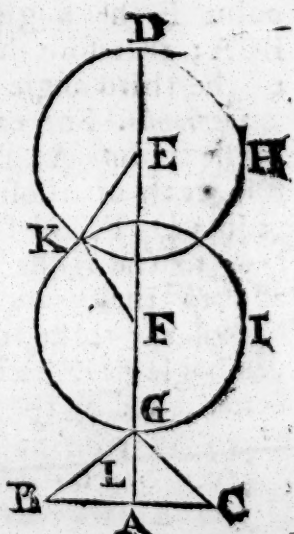
Again, Let EF be put equal to BC ; and seeing that BC is given by magnitude, EF shall be so also. But the said EF is in like manner given by position, and the point E is given: Therefore *b* the point F is given. *b 27. prop.*

Furthermore, Let FG be taken equal to AC . Now forasmuch as the said AC is given by magnitude, FG is so also. But FG is also given by position, and the point F is given: Therefore the point G is also given. Now from the center E , with the distance ED , let there be described the circle DHK , and that circle shall be given by position. *c 6. def.*

Again, on the center F , and distance FG , let there be described the circle GLK . Therefore *d* the said circle GLK is given by position; and therefore *e* the point of intersection K is given. But each of the points E and F is given: Therefore each line *f* EK , EF , and FK , is given by position and magnitude. Therefore the triangle EKF is given * by kind; but it is equal and alike to the triangle ABC ; and therefore the triangle ABC is also given by kind. *d 6. def. e 25. prop. f 26. prop.*

Scholium.

* *EUCLIDE* supposeth here, that a triangle, whose sides are given by magnitude and position, is given by kind; but the antient Interpreters demonstrate it in a manner thus. Forasmuch as the right lines KE and EF are given, *g* the reason which they have to one another is given. Also the right lines EF and FK being given, their reason is also given; and in like manner, the reason *C c*

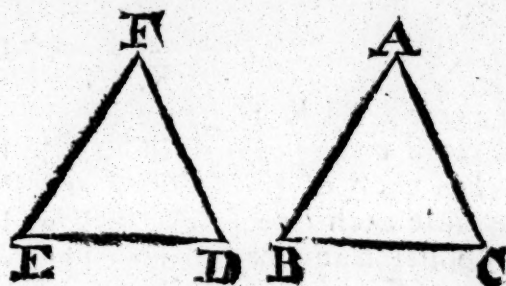


g 1. prop.

of

of the said EK and FK is given. Again, seeing that the same lines KE and EF are given by position, *h sch. 30. prop.* the angle KEF is given by magnitude: Moreover, the right lines EF and FK being given by position the angle EFK is given by magnitude, as is also the residue EKF , and so in the triangle EKF are all the angles given, and also the reasons of the sides: Therefore in the said triangle EKF is given by kind.

PROP. XL.



If the angles of a triangle ABC , are given by magnitude, the triangle is given by kind.

Constr. Let there be exposed the right line DE , given by position and by magnitude; and let there be constituted at the point D the angle EDF , equal to the angle CBA ; but in the point E the angle DEF , equal to the angle BCA ; therefore the third angle BAC is equal to the third angle DFE .

Demonstr. For each of the angles constituted in the points A , B , and C , is given: Therefore each of those which are posited in the points D , F , and E , is also given; and seeing that to the right line DE given by position, and to the point D given therein, there is drawn the right line DF , which makes the given angle EDF , *a* the line DF is given by position; and by the same reason, the line EF is given by position: Therefore *b* the point F is given by position. But each of the points *c* D and E is given: Therefore *c* each of the lines

lines DF, DE, and EF, is given by magnitude: Wherefore the triangle DFE is given by kind; and is alike to the triangle ABC: Therefore the triangle ABC is given by kind.

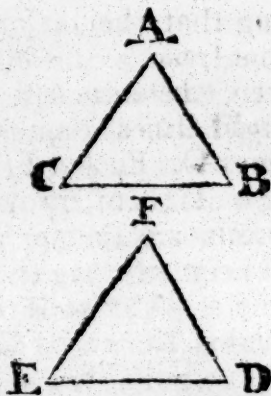
P R O P. XLI.

If a triangle ABC, hath one angle BAC given, and that the two sides BA and AC, which do constitute it, have to one another a given reason, the triangle is given by kind.

Constr. For, let there be exposed the right line DF given by magnitude and position. But thereon, and at the given point F, let there be constituted the angle DFE equal to the angle BAC.

Demonstr. Now the angle BAC is given: Therefore also the angle DFE is given, and seeing that to the right line DF given by position, and from the given point F therein is drawn a right line FE making the given angle DFE, a the said line FE is given by position. But seeing that the

reason of AB to AC is given, let the same be made of DF to FE, then let DE be drawn. Therefore the reason of DF to FE is given. But DF is given: Therefore b FE is given by magnitude. But the same FE is also given by position, and the point F is given. Therefore c the point E is also given. But each of the points D and F is given: Therefore d each of the right lines d 26. prop. DF, FE, and DE, is given by position and magnitude. Wherefore e the triangle DEF is e 39. prop. given



a 29. prop.

b 2. prop.

c 27. prop.

d 26. prop.

e 39. prop.

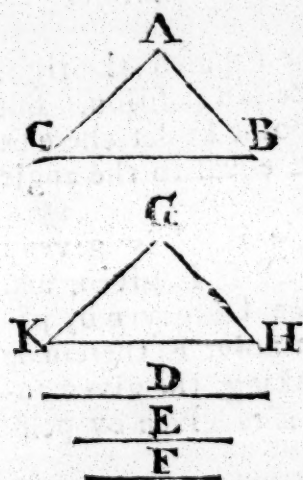
C c 2

given

f 6. 6.

given by kind. And seeing that the two triangles ABC and DEF have an angle equal to an angle, that is to say, the angle BAC to the angle DFE , and the sides which constitute those equal angles, proportional; *f* the triangle ABC is alike to the triangle DEF . But the triangle DEF is given by kind: Therefore the triangle ABC is given by kind.

P R O P. XLII.



a 2. prop.

b 2. prop.

c 39. prop.

If the sides of a triangle ABC , be to one another in given reasons, the triangle ABC is given by kind.

Constr. For, Let there be exposed the right line D , given by magnitude, and seeing that the reason of BC to AC is given, let the same be made of D to F .

Demonstr. Now D is given, therefore *a* E is also given. Again, see-

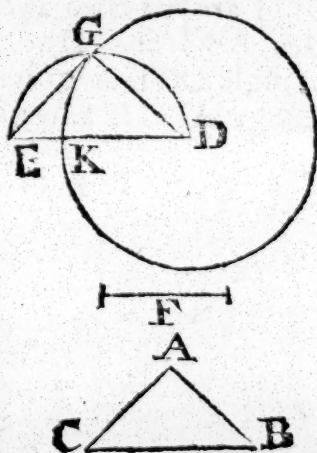
ing that the reason of AC to AB is given, let the same be made of E to F . Now E is given, therefore *b* F is also given. Now of three right lines, equal to the three given right lines D , E , and F , (and of which three lines, two of them, in what manner soever they be taken, are greater than the other,) let there be constituted the triangle GKH , in such sort as D may be equal to HK ; but E is equal to KG , and GH equal to F ; therefore each of the said lines HK , KG , and GH , is given by magnitude: Wherefore *c* the triangle HKG is given by kind. And seeing that as BC is to CA , so is D to E , and that D is equal to HK , and E to KG , as BC is to CA , so HK is to KG .

KG. Again, seeing that as CA is to AB, so is E to F, and that E is equal to KG, and F to GH; as CA is to AB, so is KG to GH. But it hath been demonstrated, that as BC is to CA, so is HK to KG: Therefore by reason of equality, as BC is to AB, so is HK to GH. Therefore *d* the triangle ABC is also *d* 5. 6. given by kind.

P R O P. XLIII.

If the sides BC and BA, about one of the acute angles of a rectangled triangle ABC, have to one another a given reason that triangle is given by kind.

Constr. Let there be exposed the right line DE given by magnitude and position, and on it let there be described the semicircle DGE: Therefore *a* the semicircle DGE is given by position.



a 6. def.

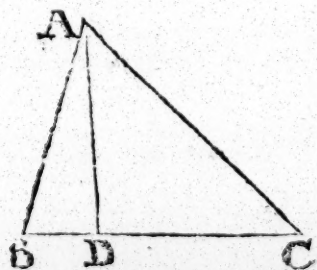
Demonstr. For the line DE being given, and divided in two equal parts, the center of the said circle is given by position, and the semidiameter by magnitude. And forasmuch as the reason of BC to BA is given, let the same be made of DE to F: Therefore the reason of DE to F is given. But DE is given, therefore F *b* is also given. Now BC is greater than *c* AB: Therefore ED is *d* also greater than F. Let DG be fitted equal to F, and let EG be drawn; then on the center D, with the distance DG, let the circle GK be described. Now that circle *e* is given by position, seeing *e* 6. def. that the center D is given, and the semidiameter

C c 3

meter

meter DG also given by magnitude, But the semicircle DGE is also given by position:
f 25. prop. Therefore *f* the point of intersection G is given,
 But the points D and E are also given, there-
g 26. prop. fore *g* each of the right lines DE, DG, and EG, is given by position and magnitude,
h 39. prop. Wherefore *h* the triangle DGE is given by kind. And seeing that the triangles ABC and DGE have an angle equal to an angle, to wit, the right angle BAC to the right angle i DGE, and the sides about the angles CBA and EDG proportional. But each of the others ACB and DEG are less than a right angle: Those triangles ABC and DEG *k* are alike. But the triangle DGE is given by kind: Therefore the triangle ABC is also given by kind,

P R O P. XLIV.



If a triangle ABC, hath one angle B given, and that the sides BA and AC, about another angle BAC, have to one another a given reason, the triangle ABC is given by kind.

Constr. Now the given angle B is either acute or obtuse, (for it was a right angle in the foregoing Proposition.) Let it be in the first place acute, and from the point A let AD be drawn perpendicular to BC.

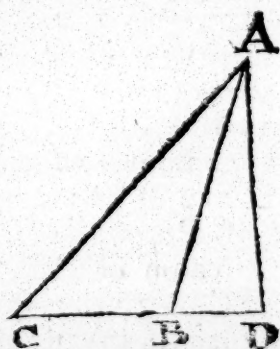
Demonstr. Therefore the angle ADB is given: But the angle B is also given; and therefore the third angle BAD is given: Wherefore
a 40. prop. *a* the triangle ABD is given by kind; and
b 3. def. therefore *b* the reason of BA to AD is given. But the reason of the same BA to AC is also given: Therefore *c* the reason of AD to AC is given, and the angle ADC is a right angle:

Where

Wherefore the triangle *d* ACD is given by *d* 43. *prop.*
 kind: Therefore *e* the angle C is given. But *e* 3. *def.*
 the angle B is also given; and therefore the
 other angle BAC is given: Therefore *f* the *f* 40. *prop.*
 triangle ABC is given by kind.

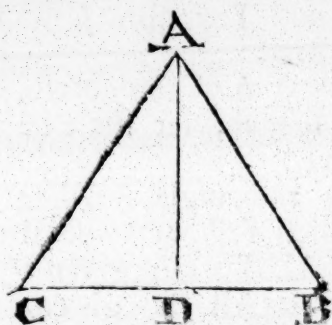
Constr. Now let the angle ABC be obtuse,
 and on the side CB prolonged, let there be
 drawn the perpendicular AD.

Demonstr. Forasmuch as the angle ABC is
 given, the angle ABD which follows it, shall
 be given. But the angle ADB is also given:
 Therefore the third angle DAB is given.
 Wherefore *g* the angle ABD is given by kind; *g* 40. *prop.*
 and therefore *h* the reason of DA to AB is given. *h* 3. *def.*
 But the reason of AB to AC is also given: There-
 fore *i* the reason of DA to AC is given, and the an-
 gle D is a right angle. Therefore the triangle
 DAC is given by kind, and therefore the angle
 ACB is given. But the angle ABC is also given:
 Therefore the third angle BAC is given.
 Wherefore the triangle ABC is given by kind.



i 8 *prop.*

PROP. XLV.



If a triangle ABC hath one angle BAC given, and also the line compounded of the two sides AB and AC , about the said given angle BAC , hath to the other side BC given reason, the triangle ABC is given by kind.

Constr. For, let the angle BAC be divided into two equal parts by the line AD , therefore

a 7. prop. a the angle CAD is given.

Demonstr. Seeing that as AB is to AC , so b is BD to CD ; by compounding, c as the line compounded of CAB is to CA , so is BC to CD , and by permutation, as the line compounded of CAB is to CB , so is CA to CD . But the reason of the line compounded of CAB to BC is given; therefore the reason of CA to CD is also given, and the angle CAD is given. Therefore d the triangle ACD is given by kind, and therefore the angle C is given. But the angle BAC is also given. Therefore the third angle B is given. Wherefore e the triangle ABC is given by kind.

d 44. prop.

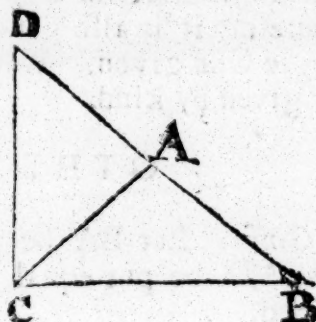
e 40. prop.

OTHERWISE.

Constr. Let BA be prolonged directly unto the point D , in such sort as that AD may be equal to AC , and let CD be joined.

Demonstr.

Demonstr. Forasmuch as the reason of the line compounded of CAB to CB is given, and that AD is equal to AC , the reason of the whole line BD to BC is given. But the angle



f 32. 1.

g 5. 1.

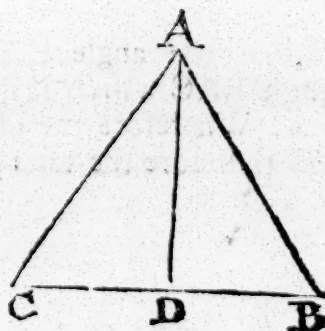
ADD is also given, for it is the half of the given angle BAC (for that the said angle BAC is equal to the two interior angles ACD and ADC , which are equal to one another being the sides AC and AD are equal:.) Wherefore the triangle BDC is given by kind, and therefore the angle B is given. But the angle BAC is also given: Therefore the remaining angle ACB is given: Wherefore the triangle ABC is given by kind.

h 44. prop.

i 40. prop.

P R O P. XLVI.

If a triangle ABC hath one angle B given, and that the line CAB compounded of the two sides CA and AB , about another angle BAC hath to the other side BC a given reason, the triangle ABC is given by kind.



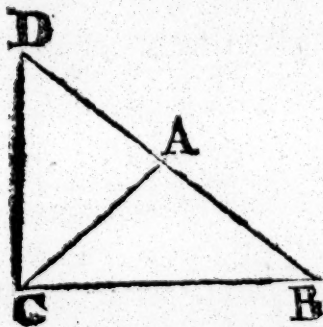
Constr. For let the angle BAC be divided into two equal parts, by the line AD .

Demonstr. Therefore (as hath been shewn in the foregoing Proposition) the compound line CAB is to CB , as AB is to BD . But the reason of the said compound line CAB to CB is given: Therefore also the reason of AB to BD is

is given. But the angle B is also given:
a 41. prop. Therefore the triangle AED *a* is given by kind;
b 2. def. and therefore *b* the angle BAD is given. But
 the angle BAC is double to that of BAD; and
 therefore it is also given. Therefore the third
 angle C is given. Wherefore the triangle ABC
 is given by kind.

OTHERWISE.

Constr. Let BA be prolonged directly, and
 let AD be put equal to AC, and let CD be
 joined.



c 41. prop.

d 3. def.

fore *d* the angle D is given: Therefore the
 angle BAC which is double to BDC, is also gi-
 ven: Wherefore the other angle ACB is given;
 and therefore the triangle ABC is given by kind.

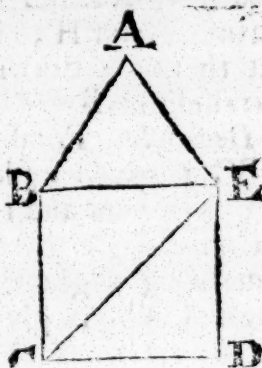
Demonstr. Forasmuch
 as the reason of the
 line compounded of
 CAB to CB is given,
 and that AD is equal
 to AC, the reason of
 BD to BC is given;
 and the angle B is also
 given: Therefore the
 triangle CDB *c* is gi-
 ven by kind; and there-

PROP.

PROP. XLVII.

Rectiline figures as $AB\text{--}CDE$, given by kind, are divided into triangles given by kind.

Constr. For let the right lines EB and EC be drawn

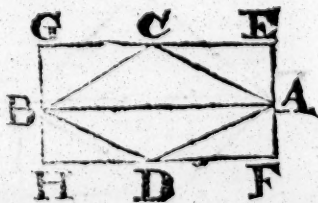


a 3. def.

Demonstr. Forasmuch as the rectiline figure $AB\text{--}CDE$ is given by kind, the angle a BAE is given, and the reason of the side AB to AE is also given: Therefore b the triangle BAE is given b 41. prop. by kind. Wherefore the angle ABE is given. But the whole angle ABC is also given: Therefore c the remaining angle EBC is given. But c 4. prop. the reason of the side AB to the side BE , and also that of AB to BC is given: Therefore d the reason of BC to BE is given, and the angle d 8. prop. CBE is also given: Therefore e the triangle BCE is given by kind. By the same discourse e 41. prop. it may be demonstrated that the triangle CDE is given by kind. Therefore the rectiline figures given by kind divide themselves into triangles given by kind.

PROP. XLVIII.

If on one and the same right line AB , are described triangles, as ACB and ABD , given by position, those triangles shall have to one another a given reason, as ACB to ABD .

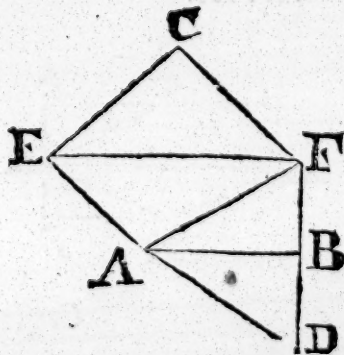


Constr.

Constr. For from the points A and B, let there be drawn at right angles on the line AB, and lines AE and BG, and prolonged unto the points F and H; but by the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

Demonstr. Forasmuch then as the triangle ABC is given by kind, *a* the reason of CA to BA is given, and the angle CAB also given; but the angle BAE is given: Therefore the remaining angle CAE is also given; but the angle CAE is given; and therefore the other angle ACE is also given. Wherefore *b* the triangle AEC is given by kind. Now the reason of EA to AB *c* is given; (for *d* the reason of EA to AC, and that of AC to AB is given;) and in like manner, the reason of FA to AB is given. Therefore *e* the reason of EA to AF is given; but as AE is to AF, so *f* the parallelogram AH to the parallelogram AG; but ACB is *g* the half of AH, and ADB the half of AG; therefore the reason of the triangle ACB to the triangle ADB is given; for it is the same reason with that of AH to AG *h*; that is to say, of EA to AF, which is given.

P R O P. XLIX.



If on one and the same right line AB there be described any two rectiline figures AECFB and ADB, given by kind, they shall have to one another a given reason (to wit) AECFB to ADB:

Constr. For let the lines FA and FE be drawn:

drawn: Therefore each of the triangles *a* ABF, *a* 47. prop. AFE, and ECF is given by kind.

Demonstr. Seeing that on one and the same right line EF there are described the triangles ECF and EAF, given by kind; the reason of ECF to EAF *b* is given. Therefore by compounding *c* the reason of AECF to EAF is given. But the reason of the said EAF to FAB is given, *d* being they are triangles given by kind, described on one and the same right line AF: Therefore *e* the reason of AECF to FAB is given. Wherefore by compounding *f* the reason of AECFB to FAB is given. But the reason of the same FAB to ABD *g* is given: Therefore *h* the reason of AECFB to ABD is also given.

b 48. prop.

c 6. prop.

d 48. prop.

e 8. prop.

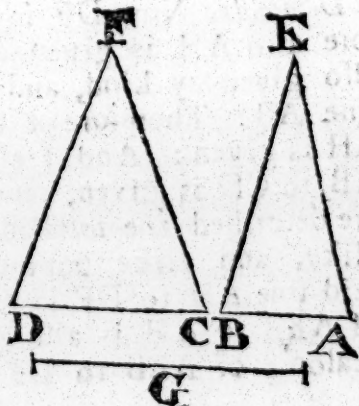
f 6. prop.

g 48. prop.

h 8. prop.

P R O P. L.

If two right lines AB and CD, have to one another a given reason, and that on those lines there be described rectiline figures AEB and CFD, alike, and alike posited, they will have to one another a given reason.



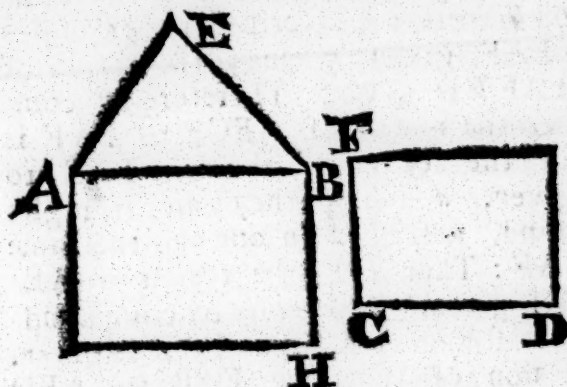
Demonstr. To the two lines AB and CD, let there be taken a third proportional: Therefore as AB is to CD, so is CD to G. But the reason of AB to CD is given: Therefore the reason of CD to G is also given: Wherefore *a* the reason of AB to G is given. But *b* as AB is to G, so is AEB to CFD: Therefore the reason of the same AEB to CFD is given.

a 8. prop.

b cor. 19.

20. 6.

P R O P.



If two right lines AB and CD have to one another a given reason, and that upon them there be described

any rectilineal figures AEB and CFD , given by kind, they will have to one another a given reason, (to wit, that of AEB to CFD .)

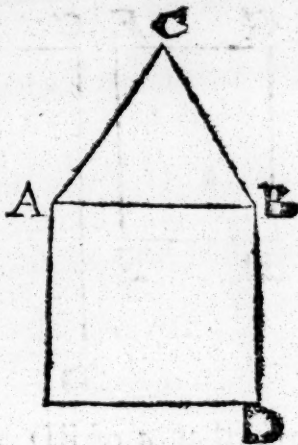
Constr. For on AB let the rectangled figure AH be described alike and alike posited to DF .

Demonstr. Now DF is given by kind: Therefore also AH is given by kind. But AEB is also given by kind, and described on the same line AB : Therefore *a* the reason of AEB to AH is given: And seeing that the reason of AB to CD is given, and that on those lines are described the rectilineal figures AH and DF alike, and alike posited, the reason *b* of the said line AH to DF is given. But the reason of AEB to AH is also given: Therefore the *c* 8. *prop.* reason *c* of AEB to DF is given.

PROP. LII.

If on a right line AB , given by magnitude, there be described a figure ACB , given by kind, that figure ACB is given by magnitude.

Constr. For on the same line AB , let the square AD be described. Therefore AD is given by kind * and by magnitude.



Demonstr. Seeing that on the right line AB , are described the two rectiline figures ACB and AD , given by kind, the reason of ACB to AD is given: Therefore ACB is given by magnitude.

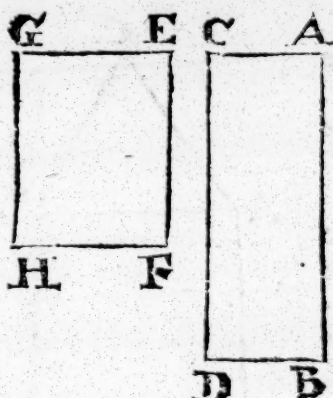
a 29. prop.
b 2. prop.

Scholium.

* The antient Interpreter hath noted here that every square is given by kind; for that all the angles thereof are given; being all equal and right angles: But also the reasons of the sides are given; for those sides being all equal, their reasons are also equal. Moreover, whensoever a square is exposed, a square equal thereto may be exhibited; and therefore the square is given by magnitude, as also each side thereof.

PROP.

PROP. LIII.



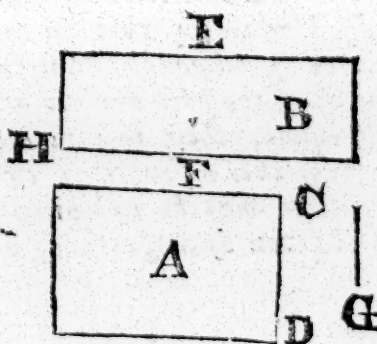
If there be two figures AD and EH, given by kind, and that one side BD of the one, hath to a side FH of the other, a given reason; the other sides shall have also to the other sides given reasons.

Demonstr. For seeing that the reason of BD to FH is given, and

a 3. def.
b 8. prop.
c 3. def.
d 8. prop.

also that *a* of BD to BA, *b* the reason of the said AB to FH is given. But the reason of the same FH to FE *c* is also given: Therefore *d* the reason of AB to EF is given. In like manner also the reasons of the other sides to the other sides are given.

PROP. LIV.



If two figures A and B, given by kind, have to one another a given reason, also their sides shall be to one another in a given reason.

Constr. For either the figure A is alike and alike posited to B, or is not: Let it in the first place be alike, and alike posited; and let there be taken the line G a third proportional to the lines CD and EF.

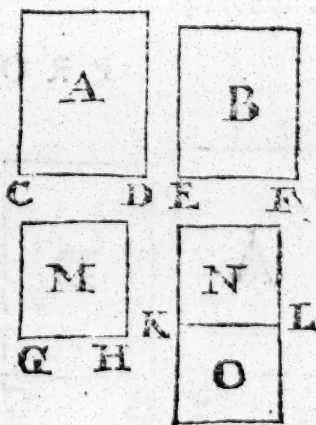
Demonstr.

Demonstr. As CD is to G, *a* so is A to B. *a* cor. 19,
 But the reason of A to B is given ; therefore 20. 6.
 also the reason of CD to G is given. And
 seeing that CD, EF, and G, are proportional,
b also the reason of CD to EF is given. But *b* 24. prop.
 A and B are given by kind : Therefore *c* the *c* 53. prop.
 other sides shall have given reasons to the other
 sides.

Now let the figure A be not alike to the
 figure B, and let there be described on EF the
 figure EH, alike and alike posited to A : There-
 fore the figure EH is given by kind ; but the *d* 49. prop.
 figure B is also given by kind : Therefore *e* the *e* 8. prop.
 reason of B to EH is given ; and therefore the
 reason of A to the same EH *e* is also given :
 But A is alike to EH : Therefore (by what is
 abovesaid) the reason of CD to EF is given ;
 and in like manner the reason of the other sides
 to the other sides is given.

OTHERWISE.

Constr. Let there be
 expoed the given line
 GH : Now either the
 figure A is alike to the
 figure B, or not. Let
 it in the first place be
 alike, and let it be as
 CD is to EF, so is GH
 to LK ; then on GH
 and LK let the figures
 M and N be described
 alike, and alike posited
 to the said A and B,
 which figures M and N shall be consequently
 given by kind.



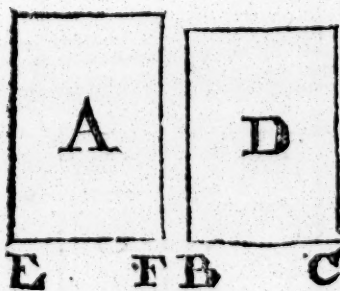
Demonstr. Therefore seeing that as CD is to
 EF, so is GH to LK. and that on those lines
 CD, EF, GH, and LK, are described the figures
 D d A, B,

f 22. 6. A, B, M, and N, alike and alike posited; *f* as A is to B, so is M to N. But the reason of A to B is given: Therefore the reason of M to N is given. But *g* M is given, considering that it is a figure given by kind, described on a right line given by magnitude; therefore N is also given.

h sch. 52. *Constr.* 2. Now, on LK let the square O be described: Therefore *h* the figure O is given by kind.

Demonstr. 2. Wherefore the reason of O to N is given. But N is given: Therefore O is given; and consequently *i* also KL. But GH is given: Therefore *k* the reason of GH to KL is given. But as GH is to LK, so is CD to EF. Therefore the reason of CD to EF is given; and therefore the figures A and B being given by kind, *l* the other sides of the same figures shall also have to the other sides given reasons. But if the figures be not alike, the latter part of the demonstration here above must be observed.

P R O P. LV.



If a space A be given by kind, and by magnitude, the sides thereof shall be given by magnitude.

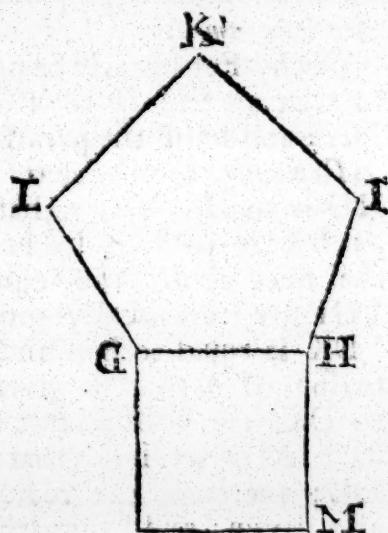
Constr. For, let the right line BC, given by position and by magnitude, be exposed; and thereon let there be described the space D, alike and alike posited to A; therefore the said space D is given by kind.

Demonstr. For that it is described on the line BC, given by magnitude, it is also *a* given by magnitude. But the figure A is also given: There-

Therefore *b* the reason of *A* to *D* is given. *b* 3. *prop.*
 But those figures *A* and *D* are given by kind :
 Therefore *c* the reason of the line *EF* to the *c* 54. *prop.*
 line *BC* is given. But *BC* is given : Therefore
d *EF* is also given. But the reason of the *d* 3. *def.*
 same *EF* to *FG* is given : Therefore *e* *FG* is *e* 2. *prop.*
 given. And by the same reasons it may be
 demonstrated that each of the other sides are
 given by magnitude.

OTHERWISE.

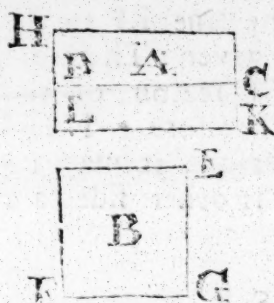
Constr. Let the
 space *GHIKL* be
 given by kind and
 by magnitude : I
 say that the sides
 thereof are given
 by magnitude. For
 on the right line
GH let there be
 described the square
GM ; therefore *f*
GM is given by
 kind.



f *sch.* 52.
prop.

Demonstr. But the
 space *GHIKL* is
 also given by kind :
 Therefore *g* the reason of the same space *GK* *g* 49. *prop.*
 to *GM* is given. But *GK* is given by magni- *h* 2. *prop.*
 tude : Therefore *h* *GM* is also given by magni- *h* 2. *prop.*
 tude ; and seeing that *GM* is the square of the
 line *GH*, *i* that line *GH* is given by magni- *i* *sch.* 52.
 tude. Wherefore in like manner, each of the *prop.*
 other lines *HI*, *IK*, *KL*, and *LG*, is given.

PROP. LVI.



If two equiangled parallelograms *A* and *B*, have to one another a given reason, as one side *CD* of the first *A*, is to one side *FG*, of the second *B*; so the other side *GE*, of the second *B*, is to that to which *DH* the other side of the first *A*, hath the

given reason that the parallelogram *A* hath to the parallelogram *B*.

Constr. For let *HD* be prolonged directly to *L*, so that as *CD* is to *FG*, so *HD* may be to *DL*; and finish the parallelogram *DK*.

Demonstr. Seeing that as *CD* is to *FG*, so *HD* is to *DL*, and *a* that *CD* is equal to *KL*; as *LK* is to *FG*, so is *GE* to *DL*; and thus the sides about the equal angles *DLK* and *EGF* are reciprocally proportional: Wherefore *b* *DK* is equal to *B*; and therefore seeing the reason of *A* to *B* is given, and that *B* is equal to *DK*, the reason of *A* to *DK* is given. But as *c* *A* is to *DK* (that is to *B*) so is *HD* to *DL*; therefore the reason of *HD* to *DL* is also given: and seeing that as *CD* is to *FG*, so *GE* is to *DL*, and that the right line *HD* hath to *DL* a given reason; to wit, that which the space *A* hath to the space *B*; as *CD* is to *FG*, so *GE* is to that to which *HD* hath the given reason that the space *A* hath to the space *B*, that is to say, the reason of *HD* to *DL*.

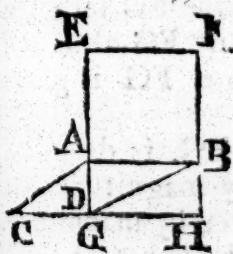
PROP.

PROP. LVII.

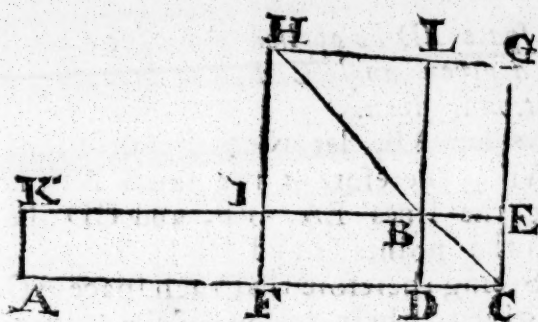
If a given space AD be applied to a given right line AB in a given angle CAB , the breadth CA of the application is given.

Constr. For on AB , let there be described the square AF ; therefore *a* the same AF is given: Let the lines EA , FB , and CD , be prolonged to the points G and H . *a sch. 52. prop.*

Demonstr. Seeing therefore that each space AD and AF is given, their reason is also given. But *b* AD is equal to AH : Therefore the reason of AF to AH is given: Wherefore the reason of EA to AG is given. (for *c* it is the same with that of AF to AH .) But EA is equal to AB ; therefore the reason of AB to AG is given. Now seeing that the angle CAB is given, and the angle GAB also given, the residue CAG is given. But the angle CGA is also given, being a right angle: Therefore the remaining angle ACG is given. Wherefore the triangle *d* CAG is given by kind. Therefore the reason of CA to AG is given. But the reason of AB to the same AG is also given; Therefore the reason of CA to AB is given; and the said AB is given: Wherefore CA is also given. *b 36.1. c 1. 6. d 40. prop.*



PROP. LVIII.



If a gi-
ven space
AB, be
applied to
a given
right line
AC, want-
ing by a fi-
gure DE,
given by

kind, the breadths of the defects are given.

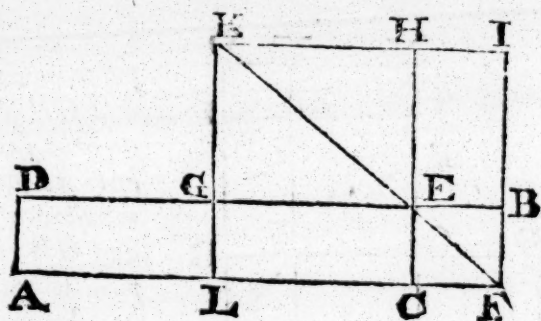
Constr. For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the said line FC let there be described the rectangled figure FG alike and alike posited to DE. Therefore FG is given by kind.

Demonstr. Seeing the figure FG is described on the right line FC given by magnitude, the said rectiline FG is *a* also given by magnitude.
a 52. prop. But FG is equal to AB and IL; (for *b* AI and FE being equal, and *c* FB and BG also equal, the Guomon ICL is equal to AB; and therefore their added figure IL common to both, FG shall be equal to AB and IL:)
b 26. 1. Therefore the figures AB and IL together are given by magnitude. But AB is given by
c 43. 1. magnitude: Therefore *d* the remaining figure IL is also given by magnitude. But it is also
d 4. prop. given by kind, seeing it is *e* alike to DE:
e 24. 6. Therefore *f* the sides of the same IL are given: Wherefore IB is given; and seeing
f 55. prop. that it is equal *g* to FD, the same FD is also given. But FC is given, therefore the remainder DC *h* is given; and *i* in a given
g 34. 1. reason to BD, and therefore *k* BD is given.
h 4. prop.
i 3. def.
k 2. prop.

PROP.

PROP. LIX.

If a given space AB be applied according to a given right line AC, exceeding it by a figure CB given by kind, the breadths of the excesses CE and CF are given.



Constr. For DE being divided into two equal parts in G, let there be described on GE the rectiline figure GH, alike and alike posited to CB.

Demonstr. Now seeing that CB is alike to GH, those figures CB and GH * are about one and the same diameter, and GH is given by kind, as is CB. But it is described on the given line GE: Therefore a the same GH is also given by magnitude. But AB is given: Therefore AB and GH are given by magnitude. Now those figures AB and GH, are equal to LI, (for AG, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI shall be equal to AB and GH;) therefore LI is given by magnitude; but it is also given by kind, being it is b alike to CB. Therefore the sides of the said LI are given, seeing it is equal to GE: Therefore d the remainder CF is given, and in a given reason e to CE. Wherefore f CE is given.

a 32. prop.

b 24. 6.

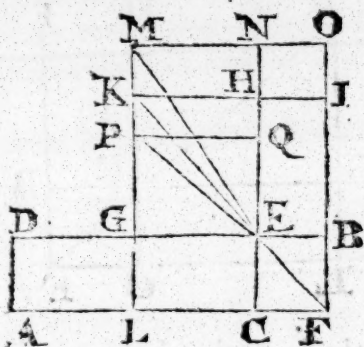
c 55. prop.

d 4. prop.

e 3. def.

f 2. prop.

Scholium.



* EUCLIDE sup-
poseth here that CB
and GH are about one
and the same diameter,
but we shall thus de-
monstrate it: Let CB
and GH be two alike
parallelograms disposed
as above, that is to
say, that the equal
angles join together in

E, the side CE meets directly with his homologal
side EH, and the side BE, his correspondent side
EG; and let the diameter FE be drawn, I say
that the said diameter FE prolonged, will pass by
the point K; that is to say, the parallelograms GH
and CB, consist about one and the same diameter.
For if it be denied, the diameter EF being pro-
duced, will pass above the point K, or below it.
Let it in the first place pass above it, and let it cut
GK, prolonged in the point M, and by the point M,
let there be drawn MN, parallel to KH, which
shall meet EH, prolonged in the point N, and FB
in O.

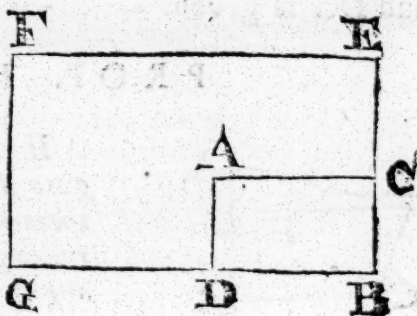
Demonstr. Forasmuch as the parallelograms
GN and CB are with the parallelogram LO
about one and the same diameter, they are
alike to one another. Wherefore as FC is to
CE, so is EG to GM. In like manner, seeing
the parallelograms CB and GH are alike, as
FC is to CE, so is EG to GK: Therefore
as EG is to GM, so is EG to GK. Where-
fore GM and GK are equal, a part to the
whole, which is absurd: By the same rea-
sons it may be demonstrated, that the diame-
ter prolonged will not fall below the point K:

There-

Therefore the parallelograms CB and GE consist about one and the same diameter.

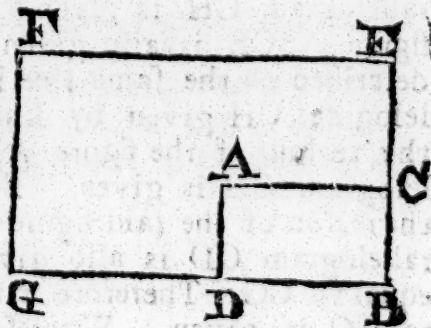
PROP. LX.

If a parallelogram AB, given by kind and by magnitude, be augmented or diminished by a Gnomon CFD, the breadths of the Gnomon (consisting of the lines CE and DG) are given.



Demonstr. For seeing that AB is given, and the Gnomon CFD also given, the whole parallelogram BF is given: But it is also given by kind, seeing it is alike to BA: therefore the sides of the same BF are given; and therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is also given. a 55. prop.

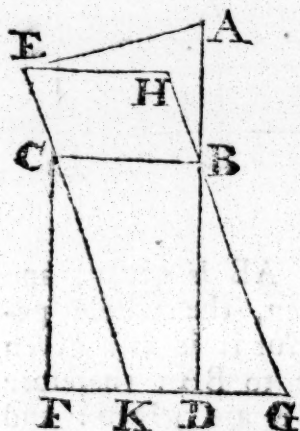
Constr. Now let the parallelogram BF, given by kind and by magnitude, be diminished by the given Gnomon CFD: I say that each of the lines CE and DG is given.



Demonstr. 2. For seeing that BF is given, and the Gnomon CFD given, the remaining figure AB is also given. But it is also given

by kind, seeing it is alike to BF: Therefore
b 55. prop. *b* the sides of the said AB are given, and therefore each of the lines CB and BD is given: But each of the lines BE and BG is given: Therefore also each of the remaining lines CE and DG is given.

PROP. LXI.



If to one side of a figure ABCE, given by kind, there be applied a space parallelogram CD in a given angle BCF, and that the given figure AC bath to the parallelogram CD a given reason, the parallelogram CD is given by kind.

Constr. For by the point B, let BH be drawn parallel to CE, and by the point E let EH be drawn parallel to CB, and let EC and HB be prolonged to the points K and G.

Demonstr. Forasmuch as the angle BCE is given, and the reason of EC to CB, *a* the parallelogram CH is given * by kind. But the figure ABCE is also given by kind, and is described on the same line BC, as the parallelogram CH given by kind is: Therefore *b* the reason of the figure ABCE to the parallelogram CH is given. But by supposition, the reason of the said figure ABCE to the parallelogram CD is also given; and CD is *c* equal to CG: Therefore *d* the reason of CH to CG is given. Wherefore the reason of the line EC to the line CK is given; (for *e* as CH is to CG, so is EC to CK.) But the reason of EC to CB is also given: Therefore *f* the reason of the said CB to CK is given. And

a 3. def.
b 49. prop.
c 26. 1.
d 8. prop.
e 1. 6.
f 8. prop.

And seeing that the angle ECB is given, also the following angle BCK *g* is given. But the *g* 13. 1. & angle BCF is proposed given; and therefore 4. *prop.* the remaining angle FCK is given. Also the angle CKF is given, for that *b* it is equal *h* 29. 1. to the angle BCK: Therefore the other angle CFK is given: Wherefore *i* the triangle FCK *i* 40. *prop.* is given by kind; and therefore the reason of FC to CK is given. But the reason of CB to the same CK is also given: Therefore *k* 8. *prop.* the reason of FC to CB is given; and the angle BCF is also given. Wherefore the parallelogram CD is given by kind.

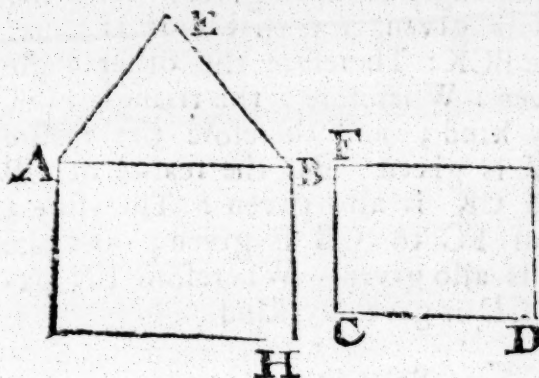
Scholium.

** Altho' it be manifest that a parallelogram that hath one angle given, and the reason of the sides about the same angle also given, is given by kind, as Euclide doth here declare, so it is notwithstanding that the antient interpreter doth thus demonstrate it.*

Seeing that in the parallelogram CH the angle ECB is given, the angle CEH is also given; for the right line EC falling on the parallels EH and CB, doth make the two internal angles on the same part equal to two right angles. And therefore seeing that the angle ECB is given, the other angles are given; and seeing that the reason of EC to CB is given, and that BH is equal to CE, and EH to BC, the reason of the sides to one another is also given.

P R O P.

PROP. XLII.



If two
right lines
AB and
CD, have
to one
another a
given rea-
son, and
that on
one of
them AB,
there be

described a figure AEB, given by kind; but on the other CD, a space parallelogram DF in a given angle DCF, and that the figure AEB hath to the parallelogram DF a given reason, the parallelogram DF is given by kind.

Constr. For on the line AB let there be described the parallelogram AH, alike and alike posited to DF.

a 50. prop.

b 8. prop.

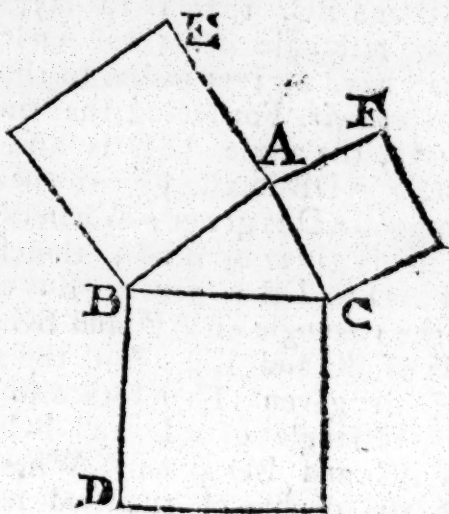
c 61. prop.

Demonstr. Seeing then that the reason of AB to CD is given, and that on those lines are described the rectiline figures AH and FD, alike and alike posited, a the reason of AH to FD is given. But the reason of FD to AEB is also given: Therefore b the reason of AH to AEB is given. But the angle ABH is also given, being equal to the angle FCD, and so the figure AEB is given by kind; and to c 61. prop. AB one of the sides thereof, the parallelogram AH is applied in a given angle ABH, and the reason of the said figure AEB to the said parallelogram AH is given: Therefore c the parallelogram AH is given by kind; and therefore FD which is alike thereto, is also given by kind.

PROP.

P R O P. LXIII.

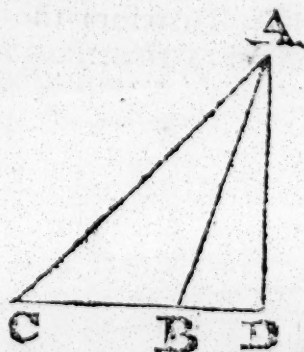
If a triangle ABC be given by kind, the square BE , CD , and CF , which is described on each of the sides, shall have a given reason to the triangle ABC .



Demonstr. For seeing that on one and the same right line BC , there are described the two rectiline figures ABC and CD , given by kind, a the reason of the same a 49. prop. ABC to CD is given; and therefore the reason of the squares BE and CF , to the triangle ABC , is also given.

P R O P. LXIV.

If a triangle ABC , bath an obtuse angle ABC given, that space of which the side AC subtending the obtuse angle ABC , is more in power than the sides AB and BC , that comprehend the said angle, shall have a given reason to the triangle AB .



Constr. Let the line CB be prolonged directly, and from the point A let the perpendicular AD be drawn: I say

a 12. 2.

say that the space of which the square of the line AC doth exceed the squares of the lines AB and BC, that is to say, *a* the double of the rectangle contained under CB and BD, shall have a given reason to the triangle ABC.

b 40. prop.

c 3. def.

d 1. 6.

e 41. 1.

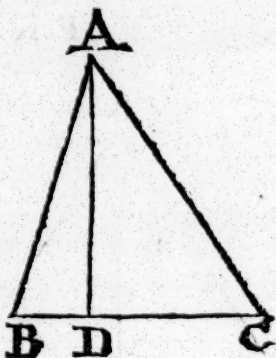
f 8. prop.

Demonstr. For seeing that the angle ABC is given, the angle ABD is also given. But the angle ADB is also given; therefore the other angle BAD is given: Wherefore *b* the triangle ABD is given by kind; therefore *c* the reason of AD to DB is given. But as AD to DB, so *d* the rectangle of AD and BC is to the rectangle of BC and BD. But the reason of AD to BD is given: Therefore also is the reason of the rectangle of AD and BC to the rectangle of BC and BD given: Wherefore the reason of the double of the said rectangle BC and BD to the rectangle of AD and BC is also given. But the said rectangle of AD and BC hath also a given reason to the triangle ABC (to wit, double reason; for the rectangle is *e* double to the triangle) therefore the reason of the double of the rectangle of BC and BD *f* to the triangle ABC is given. But the same double of the rectangle of CB and BD is that space of which the square of the line AC doth exceed the squares of the lines AB and BC: Therefore the same space hath a given reason to the triangle ABC.

PROP

PROP. LXV.

If a triangle ABC , hath one acute angle ADB given, that space, of which the side subtending the said acute angle is less in power than the sides comprehending the same acute angle, shall have a given reason to the triangle.

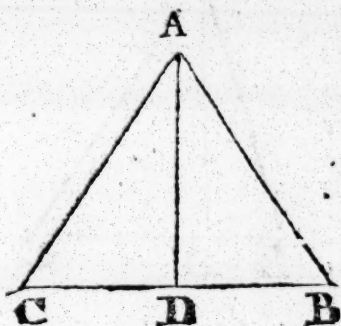


Constr. From the point A let there be drawn the line AD, perpendicular to BC: I say, that space of which the square of the line AB is less than the squares of the lines AC and CB, that is to say, a the double of the rectangle of BC and CD, hath a given reason to the triangle ABC. a 13. 2.

Demonstr. For seeing that the angle C is given, and the angle ADC also given, the other angle DAC is given: Wherefore the triangle ADC is given by kind; and therefore the reason of AD to DC is given, and consequently also that of the rectangle of BC and CD to the rectangle of BC and AD: b 40. prop.
Therefore the reason of the double of the rectangle of BC and CD to the rectangle of BC and AD is given. But the reason of the same rectangle of BC and AD to the triangle ABC is given (for the rectangle is double to the triangle: c 1. 6.) Therefore the reason of the double of the rectangle of BC and CD to the triangle ABC is given. And seeing that the same double of the rectangle of BC and CD is that whereof the square of the line AB is less than the squares of the lines AC and BC, that space of which the square of the line AB d 41. 1.
e 8. prop.

AB is less than the squares of the lines AC and BC, shall have a given reason to the triangle ABC.

PROP. LXVI.



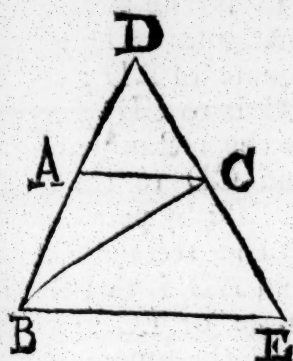
If a triangle ACB, hath one angle B given, the rectangle made of the lines AB and BC, containing the same angle, shall have a given reason to the triangle.

Constr. For from the point A let AD be drawn perpendicular to CB.

Demonstr. Therefore seeing that the angle B is given, and also the angle ADB; the other angle BAD is likewise given. Wherefore the triangle ADB *a* is given by kind; and consequently the reason of AB to AD is given. But as AB is to AD, *b* so the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the reason of the rectangle of AC and CB to the rectangle of CB and AD is given. But the reason of the said rectangle of CB and AD to the triangle ACB is also given; (for that it is double reason, the rectangle being double *c* to the triangle :) Therefore *d* the reason of the rectangle of AC and CB to the triangle ABD is given.

PROP.

P R O P. LXVII.



If a triangle *ABC* hath one angle *BAC* given, that space by which the square of the line compounded of the two sides *BA* and *AC*, that contain the same given angle *BAC* doth exceed the square of the other side, shall have a given reason to the triangle *ABC*.

Constr. For let *BA* be prolonged in such sort as that *AD* may be equal to *AC*, then having drawn *DCE* infinitely, from the point *B* let *BE* be drawn parallel to *AC*, meeting the said *DE* in the point *E*.

Demonstr. Forasmuch as *AD* is equal to *AC*, a *DB* is equal to *BE*; (for the two triangles *ADC* and *BDE* are alike) and from the top *B* is drawn to the base *DE*, the right line *BC*: Therefore * the rectangle of *DC* and *CE*, with the square of *BC*, is equal to the square of *BD*; but the same *BD* is compounded of *BA* and *AC*; therefore the square of the compound of *AB* and *AC* is greater than the square of *BC*, of the rectangle of *DC* and *CE*. a 4. 6 & 14. 5.

Now I say that the rectangle of *DC* and *CE* hath a given reason to the triangle *ABC*: Forasmuch as the angle *BAC* is given, the angle *DAC* is also given. But each of the angles *ADC* and *ACD* is given, it being the half of the angle *BAC* which is given. Therefore b the triangle *ADC* is given by kind; and therefore the reason of *DA* to *DC* is given. Therefore c the reason of the square of the said *DA* to the square of *DC* is also given. b 40. prop. c 50. prop.

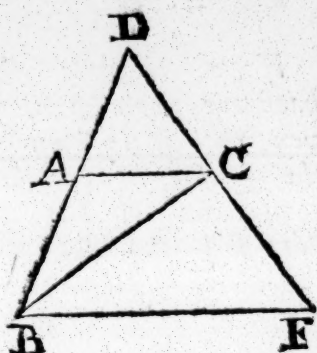
E e

given.

d 2. 6.

e 1. 6.

f 1. 6.



given. And seeing that as BA is to AD, *d* so is EC to CD, and also as BA is to AD, *e* so is the rectangle of BA and AD to the square of AD; and as EC is to CD, *f* so also is the rectangle of EC and CD to the square of CD; by permutation, as the rectangle of BA and AD is to the rectangle of EC and CD,

so is the square of AD to the square of DC. But the reason of the said square of AD to the square of DC is given: Therefore the reason of the rectangle of BA and AD to the rectangle of EC and CD is also given. But AD is equal to AC: Therefore the reason of the rectangle of BA and AC to the rectangle of EC and CD is given. But the reason of the rectangle of BA and AC to the triangle ABC *g* is given, because the angle BAC is given: Therefore *h* the reason of the rectangle EC and CD to the triangle ABC is given. But the rectangle of EC and CD is that whereof the square of the line compounded of BA and AC is greater than the square of BC: Therefore that space by which the square of the line compounded of BA and AC is greater than the square of BC, shall have a given reason to the triangle ABC.

g 66. prop.
h 8. prop.

Scholium.

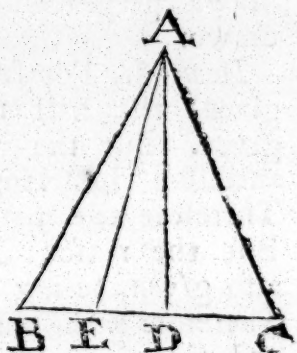
* EUCLIDE supposeth in this place, that when in an Isosceles triangle a right line is drawn from the top to the base, the square of that line, with the rectangle contained under the segments of the base,

bases, is equal to the square of either of the other legs, which the antient Interpreter doth thus demonstrate.

Constr. Let ABC be an Isosceles triangle, whose legs are AB and AC; and from the top A let AD be drawn to the base BC: I say that the square of AD with the rectangle of BD and DC, is equal to the square of either of the legs AB or AC.

Demonstr. Now the line AD is perpendicular to BD, or not: Let

it in the first place be perpendicular: Therefore it will cut the base BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the square of the said BD, and adding to them the common square of AD, the rectangle of BD and DC with the square of AD,



shall be equal to the squares of DB and AD. But to those squares of AD and DB the square of AB is equal: Therefore the square of AB is equal to the rectangle of BD and DC, and the square of AD together. i 47. 1.

Now suppose AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being so, BC shall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the square of DE, is equal to the square of BE; and $k \therefore 2$. adding the common square of AE, the rectangle of BD and DC, with the squares of DE and AE, shall be equal to the squares of BE and AE. But l the square of AD is equal to i 47. 1. the two squares of DE and AE: Therefore

E e 2

the

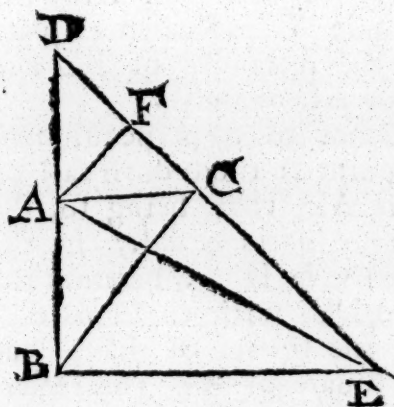
the rectangle of BD and DC, with the square of AD, is equal to the squares of BE and AE. But to these squares of BE and AE the square of AB is equal: Therefore the square of AD, with the rectangle of BD and DC, is equal to the square of AB.

O T H E R W I S E.

Constr. Having done, as in the foregoing Demonstration, from the point A, let AF be drawn perpendicular to CD, and let AE be drawn.

Demonstr. Forasmuch as the angle BAC is given, the half thereof ACF shall be also given. But the angle AFC is given; and therefore the triangle AFC is given by kind: Therefore the reason of AF to FC is given. But the reason of CD to the same FC is also given, seeing that CD is double to FC: *m* 8. *prop.* Therefore *m* the reason of CD to AF is given; and therefore also the reason of the rectangle of CD and EC, to the rectangle of AF and EC, is given; (for it is the same reason *n* as that of CD to AF.)

O 4 F. I.



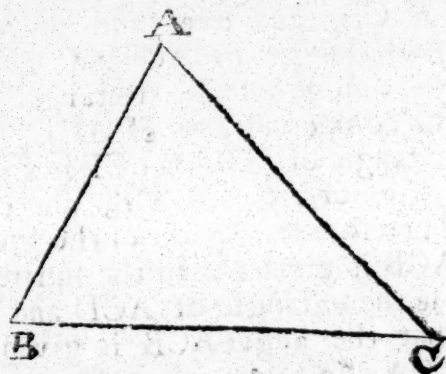
p. 37. I.

But the triangle ACE is equal to the triangle ABC *p*, they being both constituted on one and the same base AC, and between the same parallels

parallels AC and BE : Therefore *q* the reason *q* 8. *prop.* of the rectangle of CE and CD to the triangle ABC is given. But the said rectangle of CE and CD is the space by which the square of the line compounded of AB and AC, is greater than the square of BC: Therefore that space by which the square of the line compounded of AB and AC is greater than the square of BC, hath a given reason to the triangle ABC.

OTHERWISE.

For the given angle A is either a right, acute, or obtuse angle : Let it in the first place be supposed a right angle : Therefore the square of the line compounded of BAC, is



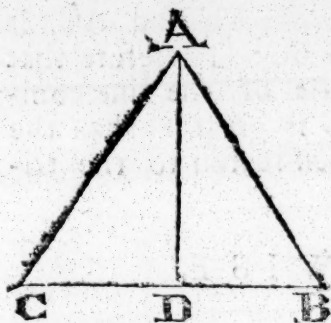
greater than the square of BC, by twice the rectangle of BA and AC ; (seeing that *r* the square of BC is equal to the squares of BA and AC ; and the square of the line compounded of BAC is equal to those two squares of BA and AC, and twice the rectangle of the said BA and AC :) Wherefore the reason of double the rectangle of BA and AC to the triangle ABC is given. *r* 47. I. *s* 5. 2.

Constr. Now let the angle C be supposed acute, and from the point A let there be drawn on CB the perpendicular AD.

Demonstr. Forasmuch as the triangle CAB is an Oxigonium triangle, and the perpendicular AD being drawn, the square of CA and CB

t 13. 5. are equal t to the square of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA

u 4. 2.



x 1. 2.

y 40. prop.

z 6. prop.

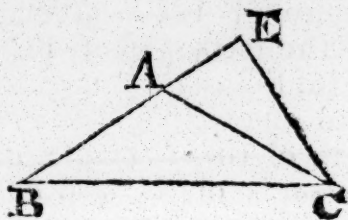
a 1. 6.

b 66. prop.

and CB, the squares of CA and CB, with the double rectangle of the said CA and CB, that is to say, u the alone square of the line compounded of ACB, are equal to the square of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to say, the double of the rectangle contained under the compound line of ACD and CB (for the rectangle of ACD and CB is x equal to the rectangles of AC and CB, and of CD and CB:) Therefore the square of the line compounded of ACB is greater than the square of AC, by double the rectangle of ACD and CB. And seeing that the angle ACB is given, and the angle BDA also given, the other angle CAD is given: Therefore y the triangle CAD is given by kind, and therefore the reason of CD to CA is given, and by consequence the reason of the line compounded of ACD to CA z is also given. Wherefore the reason of the rectangle of those lines compounded of ACD and CB a to the rectangle of AC and CB is also given. But the reason of the said rectangle of AC and CB to the triangle CAB b is given, seeing the angle C is given; therefore the reason of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.

Lastly,

Lastly, let the angle BAC be supposed to be obtuse, and having prolonged BA, from the point C, let the perpendicular CE be drawn on the said line BA prolonged; and let AF be proposed to be equal to AE.



Demonstr. Forasmuch as the angle BAC is obtuse, and the perpendicular CE being drawn, the squares of AB and AC, and the double of the rectangle under BA and AE, or AF, are all alike equal to the square of BC, and c 12. 2.

adding the common double rectangle of BA and AC, the squares of the said AB and AC, with the double of the rectangle of the same AB and AC, that is to say, d the square of d 4. 2.

the line compounded of BAC, and the double of the rectangle of BA and AF are together equal to the square of BC, with the double of the rectangle of BA and AC. Let the common double of the rectangle of BA and AF be taken away, and there will remain the square of the line compounded of BAC, equal to the square of BC, with the rectangle of AB and CF; (for the rectangle of AB and AC is equal e to the two rectangles of AB and AE, and of e 1. 2.

AB and CF:) Therefore the square of the line compounded of BAC is greater than the square of BC by the double of the rectangle of AB and CF. And forasmuch as the angle BAC is given, the angle CAE f is given. But f 13. 11. the angle AEC is also given; therefore the g 40. prop. other angle ACE is given: Wherefore g the triangle ACE is given by kind, and therefore the reason of CA to AE, that is to say, to AF is h 5. prop. given. Therefore h the reason of the said CA to FG is also given. But the reason of the same CA

CA

i 8. prop.

CA to CE is given ; therefore *i* the reason of CE to CF is also given. Wherefore the reason of the rectangle of EC and AB to the rectangle of FC and AB is given ; (for the rectangle is to the rectangle *k* as CE is to CF) and also

k 2. 6.

that of the rectangle of AC and AB to the rectangle of EC and AB. Therefore *l* the rea-

l 8. prop.

son of the rectangle of FC and AB to the rectangle of AC and AB is given. But the reason of the rectangle of AC and AB to the triangle ABC *m* is given : Therefore also the reason

m 66. prop.

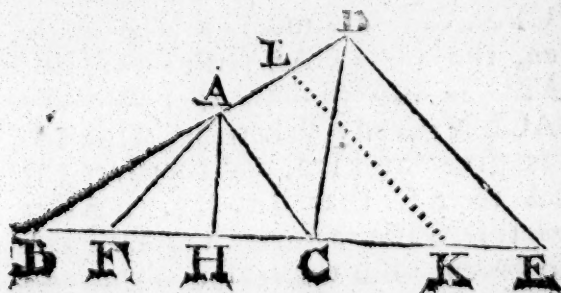
of the double of the rectangle of FC and AB, to the triangle ABC is given. But the same double of the rectangle of FC and AB is that, whereof the square of the line compounded of BAC is greater than the square of BC, wherefore that space of which the square of the line compounded of BAC is greater than the square of BC, hath a given reason to the triangle ABC.

O T H E R W I S E.

Constr. Let the line BA be prolonged to the point D, in such sort as AD may be equal to AC, and let CD be drawn.

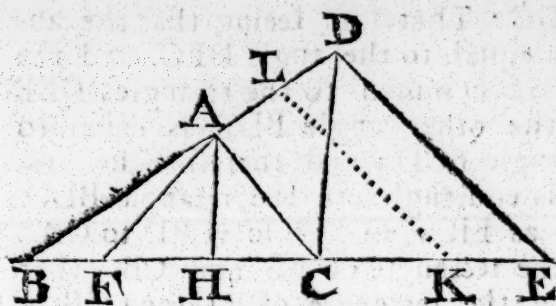
Demonstr. Forasmuch as the angle BAC is given, each of the angles ADC and ACD, which is the half thereof, shall be also given ; and therefore the other angle DAC is also given ; Therefore *n* the triangle ACD is given by kind. Wherefore the reason of AC to CD is given, And forasmuch as the an-

n 40. prop.



gle ADC is given ; Let each of the angles DEC and AFC be made equal to the

the said ADC: Therefore seeing that the angle BDC is equal to the angle DEC, and the angle DBE is common to the triangles DBE and DBC, the other angle BDE is equal to the other angle BCD; and therefore the triangle BDE is equiangled to the triangle BDC. Therefore *o* as EB is to BD, so is BD to CB: *o* 4. 6. Wherefore the rectangle of EB and CB, that is to say, *p* the rectangle of EC and CB, *q* p 5. 2. with the square of CB is equal, *r* to the square *q* 5. 2. of BD, that is to say, to the square of the *r* 17. 6. line compounded of BAC; for AD is equal to AC; and therefore the rectangle of EC and CB with the square of CB, that is to say, the square of the line compounded of BAC is greater than the square of the rectangle of BC and CE: I say therefore that the reason of the said rectangle of BC and CE to the triangle ABC is given. Forasmuch as the angle BDE is equal to the angle BCD, and the angle ADC equal to the angle ACD, the other angle CDE is equal to the other angle ACB: But the angle DEC is also equal to the angle AFC; therefore the remaining angle CAF is equal to the remaining angle DCE. Wherefore the triangle AFC is equiangled to the triangle DCE; and therefore *s* as CA is to AF, so is CD *s* 4. 6. to CE; and by permutation, as AC is to CD, so is AF to CE. But the reason of AC to CD is given: Therefore also the reason of AF to CE is given. From the point A let AH be drawn perpendicular to BC: Forasmuch as the angle AFC is given, and the angle AHF also given, the third angle HAF is given: Wherefore *t* the triangle AHF is *t* 40. *prop.* given



given by
kind ;
and by
confe-
quence
the rea-
son of
AF to
AH is
given.

But the reason of AF to CE is also given :
Therefore u the reason of AH to CE is gi-
ven ; and therefore the reason of the rectangle
of AH and BC x to the rectangle of BC and
CE is also given. But the reason of the rect-
angle of AH and BC to the triangle ABC is
likewise given ; (for the rectangle y is double
to the triangle) and the rectangle of BC and
CE is that whereof the square of the line
compounded of BAC is greater than the square
of BC. Therefore that space of which the
square of the line compounded of BAC is
greater than the square of BC by a given rea-
son to the triangle ADC.

Scholium.

† The antient Interpreter pretending to shew the
construction of the angle DEB equal to the angle
ADC, saith that on the line BD and in the point
D, the angle BDE ought to be made equal to the
angle BCD, and that the right lines BC and DE be
drawn until they intersect in E, in such sort as he
supposeth the angle BCD, to be given, but it is
not.

The same Interpreter afterward shews how there
may universally from a given point be drawn a
right line given by position to a right line, making
an angle equal to a given angle. But we will
also reject this way, seeing we have elsewhere
shewn

shewn another more brief and easy. For example, if we would from the point D draw to the line BC given by position a right line, making an angle equal to a given angle ADC , as is here required, we have no more to do but to assume the point K in the said line BC , and there make the triangle CKL equal to the given angle ADC : If the line KL doth meet with the point D , it shall be the line required. But if it meet not with it, from the point D let there be drawn the line DE , parallel to the said KL , cutting BC prolonged in E , and the angle DEC shall be equal to the given angle ADC , for on the two parallel lines LK and DE , there doth fall the line BE ; and therefore the angle DEC is equal to the angle LKC , which hath been made equal to the given angle ADC ; and by consequence the same angle DEC is also equal to ADC . z 29. 1.

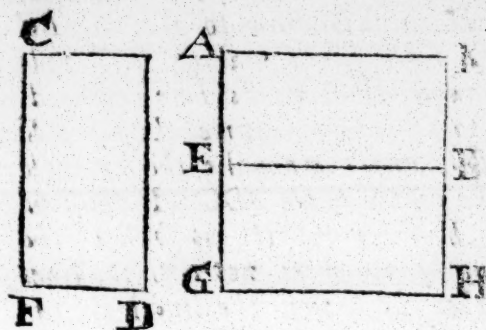
P R O P. LXVIII.

If two parallelograms AB and CD have to one another a given reason, and that a side hath also a given reason to a side, the other side shall have likewise a given reason to the other side.

Constr. Let the reason of BE to FD be given: I say the reason of AE to FC is also given. For to the right line EB let there be applied the parallelogram EH , equal to the parallelogram CD , and constituted in such sort as AE and EG may make one right line: † Therefore KB and BH will also make one right line.

Demonstr.

a 1. 6.



b 14. 6.

therefore the reason of AE to EG is also given. Seeing therefore that EH is equal and equiangled to CD, as *b* EB is to FD, so is FC to EG. But the reason of EB to FD is given: Therefore also the reason of FC to EG is given. But the reason of AE to the same EG is also given: Therefore the reason of AE to FC is given.

Scholium.

c 13. 1.

d 29. 1.

† EUCLIDE having posited AE and EG directly in one right line, presently concludeth that KB and BH shall also make a right line; but we shall demonstrate it thus. Seeing the lines AE and EG are posited directly, the angles AEB and BEG *c* are equal to two right angles; and seeing that AB is a parallelogram, the lines AK and EB are parallels, on which the line AE doth fall; and therefore the two internal angles A and BEA *d* are also equal to two right angles, and taking away the common angle BEA, there will remain the angle A, equal to the angle BEG; and consequently their opposite angles EBK and H are also equal to one another. Again, seeing that BG is a parallelogram, the two lines BE and HG are parallels, on which BH doth fall; and therefore the two internal angles H and EBH *d* are equal to two right angles. But it hath been demonstrated that

Demonstr.
Forasmuch as the reason of AB to CD is given, and that EH is equal to the said CD; the reason of AB to EH is given; and

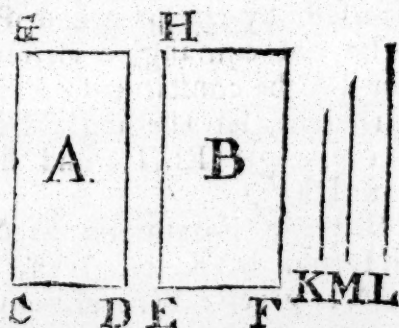
that H is equal to EBK : Therefore the two angles EBK and EBH are also equal to two right angles; and therefore the two lines KB and BH do meet e 14. 1.
directly according to EUCLIDE.

OTHERWISE.

Constr. Let the given right line K be exposed, and seeing that the reason of A to B is given, let the same be made of K to L ; therefore the reason of K to L is also given.

Demonstr. But K is given; therefore f L is also given. f 2. prop. Again, seeing that the reason of CD to EF is given, let the same be made of K to M : Therefore the reason of K to M is given. But K is given, therefore g M is also given; g 1. prop. and

therefore the reason of L to M is given. Now seeing that A is equiangled to B , the reason of the said A to B is compounded of that of the sides, that is to say, of CD

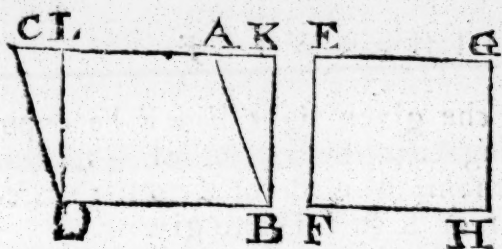


h 23. 6.

to EF , and of CG to EH . But also the reason of K to L is compounded of K to M , and of M to L ; therefore the reason compounded of CD to EF , and of CG to EH , is the same with that which is compounded of K to M , and of M to L (the reason of K to L being the same as of A to B :) But the reason of CD to EF is the same as of K to M : Therefore the other reason of CG to EH is also the same as of M to L . But the said reason of M to L is given: Therefore also the reason of CG to EH is given.

PROP.

PROP. LXIX.



If two pa-
rallelograms,
CB and EH,
having the
angles D and
F given, and
that a side
bath also a
given reason

to a side; in like manner the other side shall have
a given reason to the other side.

Constr. Let the reason of BD to FH be also
given: I say that the reason of AB to EF is
given. For if CB be equiangled to HE, it is
manifest by the precedent Proposition; but if
it be not equiangled thereto, let the right
line DB be constituted, and in the given point
B therein, let the angle DBK be made equal
to the angle EFH, and finish the parallelo-
gram DK.

Demonstr. Forasmuch as each of the angles
BKL and BAK is given, † the other angle
a 40. prop. KBA is given: Wherefore the triangle a ABK
is given by kind; and therefore the reason of
AB to BK is given. But the reason of CB to
b 35. prop. EH is supposed to be given, and b CB is equal
to DK; therefore the reason of DK to EH
is given; and seeing that DK is equiangled to
EH, and the reason of the said DK to EH is
e 68. prop. given, as also that of DB to FH, c the reason
of BK to FE is given. But the reason of the
d 29. 1. said BK to BA is also given: Therefore d the
reason of AB to FE is given.

Scholium.

Scholium.

† EUCLIDE supposeth here that a parallelogram having one angle given, all the other angles are also given, and as well the antient Interpreters as others, do give the reasons why, the angle F being given, the other angle E shall be also given, it being the remainder of two right angles, for that on the parallel lines EG and FH there doth fall the line EF , which makes e the two internal angles (of the same part) F and G , equal to two right angles. But to those angles f the opposite angles G and H are equal, and therefore they are also given.

e 29. 1.
f 34. 1.

From whence it follows that the angles BDC and F being given by supposition, all the other angles of the two parallelograms CB and EH , are also given: Therefore the angle DBK having been made equal to the angle F , the angle K shall be equal to the angle E , and given as that is: But the angle BAL , which is opposite to the given angle BDC , is also given; and therefore BAK which is the remainder of two right angles, shall be also given; in such sort as in the triangle ABK , the two angles BAK and BKA are given, as EUCLIDE doth declare in this place.

P R O P. LXX.

If of two parallelograms AB and EH , the sides about the equal angles, or about the unequal angles (yet nevertheless given angles) have to one another a given reason, to wit (AC to EF , and CB to FH) also the same parallelograms AB and EH shall have to one another a given reason.

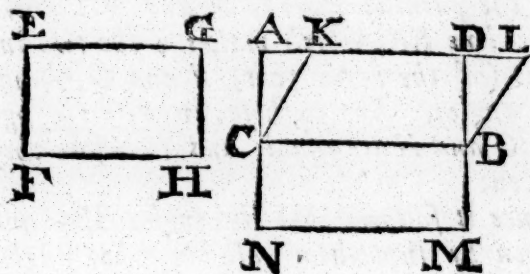
Constr. For let AB be prolonged to EH , and on the right line CB let the parallelogram CM be applied equal to the parallelogram EH , in such

a sch. 68.
prop.

b 14. 6.

such sort as AC may be direct to CN ; that is to say, that AC and CN make one right line ; and by consequence DB shall be a directly with BM. .

Demonstr. Forasmuch then as CM is equi-angled and equal to EH, the sides about the equal angles shall be reciprocally *b* proportional: Wherefore as BC is to HF, so is FE to



NC. But thereason of BC to HF is given: Therefore the reason of FE to NC is also given.

c 8. prop.

d 1. 6.

But the reason of AC to the same EF is given: Therefore *c* the reason of AC to NC is also given. Wherefore the reason of AB to CM is given; (for it is the same *d* as of AC to CN.) But CM is equal to EH: Therefore the reason of AB to EH is given.

Constr. Now suppose AB not to be equi-angled to EH, and on the right line CB, and in the given point C therein: Let there be constituted the angle BCK, equal to the given angle F, and so finish the parallelogram CL.

e 40. prop.

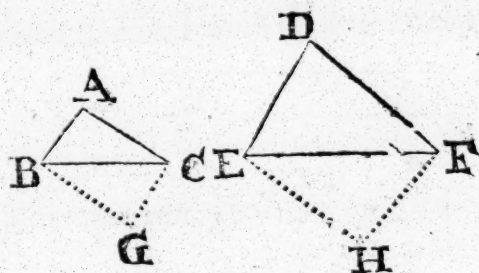
Demonstr. Forasmuch as the angle ACB is given, and the angle BCK also given, the remaining angle ACK is given: Therefore the triangle ACK *e* is given by kind: and therefore the reason of AC to CK is given: But the reason of AC to EF is also given: Therefore the reason of CK to EF is given. But the reason of BC to HF is also given, and the angle BCK is equal to the angle F; there-

fore

fore (by the first part of this Proposition) the reason of CL to EH is given. But to the said CL, AB is equal : Therefore the reason of AB to EH is given.

PROP. LXXI.

If of two triangles ABC and DEF, the sides about the equal angles A and D, or else about the unequal angles (yet neverthe-

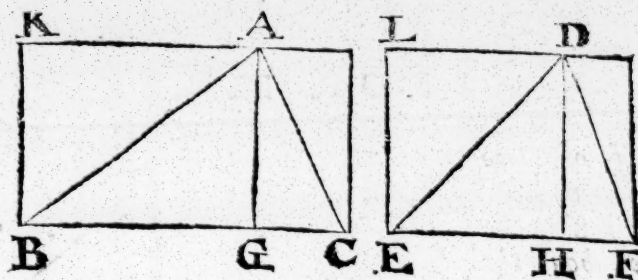


less given angles) have to one another a given reason (to wit, AB to DE and AC to DF) the same triangles shall have also to one another a given reason ABC to DEF.

Constr. Let the parallelogram AG and DH be finished.

Demonstr. Seeing that the two parallelograms AG and DH, have the sides about the equal angles A and D, or else about the unequal angles (nevertheless given) have a given reason to one another, the reason *a* of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG *b* and the triangle DEF the half *b* of the parallelogram DH. Therefore the reason of the triangle ABC to the triangle DEF is given.

PROP. LXXII.



If of two triangles ABC and DEF , the bases BC and EF , are in a given reason, BC to EF , and that from the angles A and D , there be drawn to those bases the right lines AG and DH , making the equal angles AGC and DHF , or else unequal (yet nevertheless given) which shall have to one another given reasons AG to DH , those triangles ABC and DEF shall have also a given reason to one another, to wit, ABC to DEF .

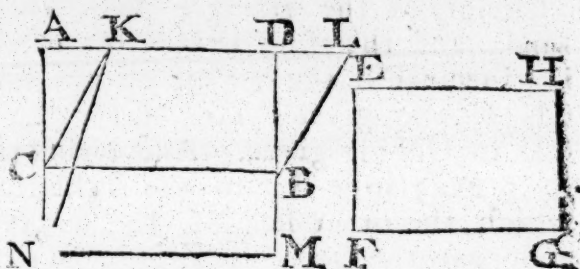
Constr. For let the parallelogram KC and LF be finished.

Demonstr. Forasmuch as the angles AGC and DHF are equal, or unequal (yet given) and that the angle AGC *a* is equal to the angle KBC . But the angle DHF equal to the angle LEF , the angles at the points B and E are equal, or else unequal (yet given) and for that the reason of AG to DH is given, and AG is equal to KB . But DH is equal to LE , also the reason of KB to LE is given. But the reason of BC to EF is also given, and the angles at the points B and E are equal, or else unequal (yet given:.) Therefore *b* the reason of the parallelogram KC to the parallelogram LF is given; and therefore the reason of the triangle ABC to the triangle DEF is given, seeing those triangles *c* are the one half of the parallelograms.

PROP.

PROP. LXXIII.

If of
two pa-
rallelo-
grams
AB and
EG, the
sides a-
bout the
equal an-



gles C and F, or else about the unequal angles (but nevertheless given) are in such sort to one another, that as the side CB of the first, is to the side FG of the second; so the other side EF of the second, is to some other right line CN. But that the other side AC, hath also to the same right line CN a given reason, those parallelograms will have also to one another a given reason AB to EG.

Constr. For in the first place, let the parallelogram AB be equiangled to EG, and having placed CN directly to AC: Let the parallelogram CM be finished.

Demonstr. Forasmuch then as CB or NM its equal, is to FG, so is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is put equal to F) the parallelograms CM and EG *a* are equal: But as *a* 14. 6. AC to CN, so *b* the parallelogram AB is to the *b* 1. 6. parallelogram CM or EG: Therefore seeing that the reason of AC to CN is given, the reason of AB to EG is also given.

Constr. 2. Now suppose the parallelogram AB not to be equiangled to the parallelogram EG, and let there be constituted at the given point C in the line CB, the angle BCK, equal to the angle EFG, and so finish the parallelogram CL.

F f 2

Demonstre

c sch. 69.

prop.

d 40. prop.

e 8. prop.

Demonstr. 2. Seeing that each of the angles ACB and KCB is given, the remaining angle ACK is also given. But *c* the angle CAK is given, as also the remaining angle AKC: Therefore *d* the triangle ACK is given by kind; and therefore the reason of AC to CK is given. But the reason of the same AC to CN is also given: Therefore *e* the reason of CK to CN is given. And seeing that as CB is to FG; so is EF to the right line CN, to which the other side KC hath a given reason, and that the angle BCK is equal to the angle F, the reason of the parallelogram CL to the parallelogram EG is given (by the first part of this Proposition) but the parallelogram CL is equal to the parallelogram AB: Therefore the reason of the parallelogram AB to the parallelogram EG is given.

P R O P. LXXIV.

If two parallelograms (as in the former figure) AB and EG, in equal angles C and F, or else in unequal angles (yet nevertheless given angles) have a given reason to one another, as one side CB of the first shall be to one side FG of the second, so the other side EF of the second, shall be to that to the which the other side AC of the first hath a given reason. (See the foregoing Scheme.)

a sch. 68.

prop.

Constr. For either AB is equiangled or not; suppose it in the first place to be equiangled, and to the right line BC let there be applied the parallelogram CM, equal to the parallelogram EG, and so posited, as that AC and CN may be direct: Therefore *a* DB and BM shall be also direct (that is, as one right line)

Demonstr. Seeing that the reason of AB to EG is given, and that CM is equal to EG, the reason of AB to CM is also given; and therefore the reason of AC to CN is given (seeing

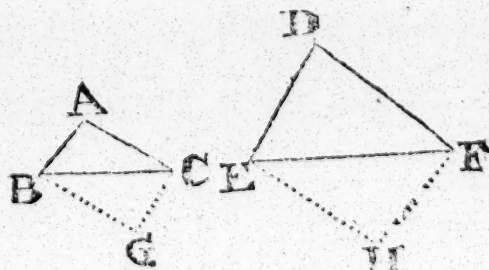
(seeing AB is to CM, *b* as AC is to CN;) and for *b* 1. 6. that CM is equal and equiangled to EG, the sides about the equal angles of the parallelograms CM and EG, *c* are reciprocally proportional; *c* 14. 6. and therefore as CB is to FG, so is EF to CN. But the reason of AC to CN is given: Therefore as CB is to FG, so is EF to that to which AC hath a given reason.

Constr. 2. Now suppose AB not to be equiangled to EG, and in the given point C of the line CB, let there be constituted the angle BCK equal to the angle EFG, and finish the parallelogram CL.

Demonstr. 2. Seeing then that the reason of AB to EG is given, and *d* that AB is equal *d* 36. 1. to CL, also the reason of CL to EG is given, and the angle BCK is equal to the angle F, and therefore CL *e* is equiangled to EG. *e* sch. 6. Therefore (by the first part of this Proposition *prop.*) as CB is to FG, so is EF to that to which CK hath a given reason. But the reason of AC to CK is given; (as appears by what hath been demonstrated in the latter part of the precedent Proposition.) Therefore as CB is to FG, so is EF to that to which AC hath a given reason.

P R O P. LXXV.

If two triangles ABC and DEF, in equal angles A and D, or else unequal (yet nevertheless given) have to one



another a given reason, as the side AB of the first, shall be to the side DE of the second, so the other

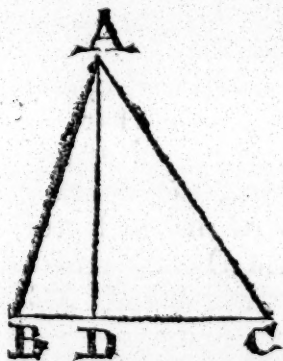
other side DF of the second, shall be to that right line to the which the other side AC of the first hath a given reason.

Constr. For let the parallelograms AG and DH be finished.

Demonstr. Forasmuch as the reason of the triangle ABC to the triangle DEF is given, also the reason of the parallelogram AG to the parallelogram DH is given.

Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (nevertheless given) have to one another a given reason; as a AB is to DE , so is DF to that to which AC hath a given reason.

P R O P. LXXVI.



If from the top A of a triangle ABC , given by kind, there be drawn to the base BC , a perpendicular line AD , that line AD shall have to the base BC a given reason.

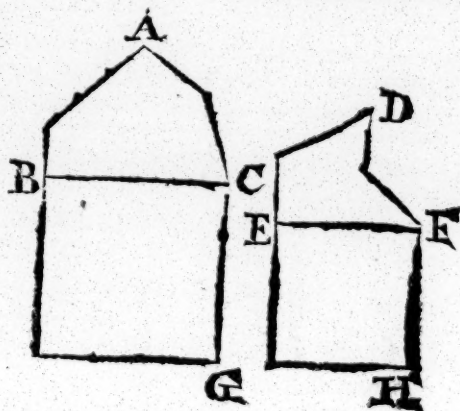
Demonstr. For seeing that the triangle ABC is given by kind, the reason of AB to BC is given; and the angle B is also given. But

the angle ADB is given; therefore the other angle BAD is given. Wherefore a the triangle ABD is given by kind; and therefore the reason of AB to AD is given. But the reason of AB to BC is given: Therefore b the reason of AD to BC is given.

PROP. LXXVII.

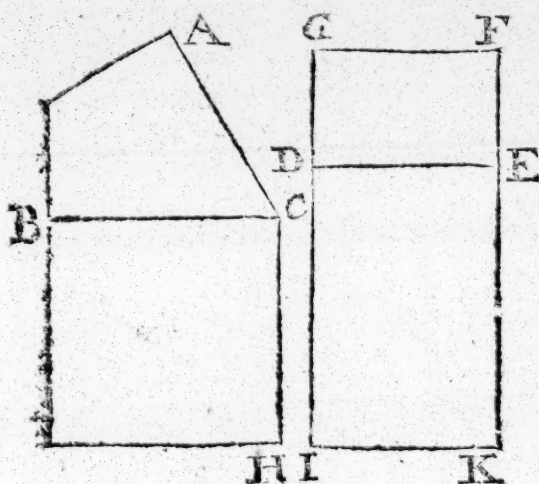
If two figures ABC and DEF , given by kind, have to another a given reason, the reason also shall be given of which you please of the sides of one of the figures, to which you please of the sides of the other figure.

Constr. For on the right lines BC and EF , let there be described the squares BG and EH .



Demonstr. Forasmuch as on one and the same right line BC are described two figures ABC and BG given by kind, *a* the reason of the said ABC to BG is given. In like manner, the reason of DEF to EH is given; and seeing that the reason of ABC to DEF is given, and also that of the same figure ABC to BG ; and again the reason of DEF to EH : *b* the reason of BG to EH *b* 8. *prop.* is given; and therefore the reason of BC to EF is also given.

PROP. LXXVIII.



If a given figure ABC , hath a given reason to some rect-angled figure DF , and that one side BC hath a given reason to one side DE , the reason.

gled figure DF is given by kind.

Constr. For on the right line BC let the square BH be described, and to the right line DE , let the parallelogram DK be applied equal to BH , in such a manner, as that GD and DI may be placed directly, *a* and by consequence FE and EK also directly.

a sch. 68. prop.

Demonstr. Therefore seeing that on one and the same right line BC are described the two rectiline figures ABC and BH , given by kind, *b* the reason of ABC to BH is given. But the reason of the said ABC to DF is also given: Therefore *c* the reason of BH to DF is given. But BH is equal to DK : Therefore the reason of DK to DF is also given. And seeing that BH is equal and equiangled to DK , both the one and the other being rect-angles, *d* the sides of those figures are reciprocally proportional; and as BC is to DE , so is DI to CH . But by supposition, the reason of BC to DE is given; therefore also the reason of DI to CH is given; but the reason

b 49. prop.

c 8. prop.

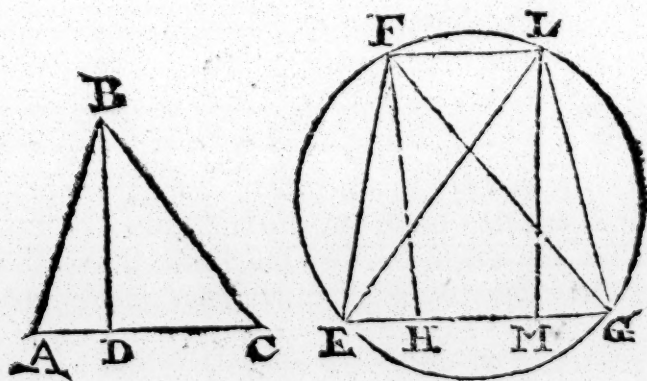
d 14. 6.

of DI to DG is also given: (for DI is to DG *e* as DK to DF:) Therefore *f* the reason of DG to CH is given. But CH is equal *f* 8. *prop.* to BC, seeing that BH is a square; therefore the reason of the same BC to DG is given: But the reason of the same BC to DE is also given; therefore the reason of DE to DG is given, and the angle at D is a right angle: Therefore *g* DF is given by kind.

g sch. 6. *prop.*

P R O P. LXXIX.

If two triangles ABC and EFG, have an angle B equal to an angle F. But from the equal angles B and F there be drawn perpendiculars BD and FH, to the bases AC and EG; and that as the base AC of the first triangle ABC, is to the perpendicular BD, so also the base EG of the other triangle EFG, is to the perpendicular FH, those triangles ABC and EFG are equiangled.



Constr. For about the triangle EFG let there be described the circle EFLG, then on the right line EG, and in the point E given therein, let there be made the angle GEL, equal to the angle C, and let FL and LG be drawn, and the perpendicular LM.

Demonstr.

a 21. 3.

b 7. 5.

c 28. 1.

d 33. 1.

e 29. 1.

f 21. 3.

Demonstr. Seeing then that the angle GEL is equal to the angle C, and the angle ELG is equal to the angle EFG, *a* they being in one and the same segment of the circle; the third angle EGL is equal to the third angle A. Wherefore the triangle ABC is alike to the triangle ELG, and the perpendiculars BD and LM are drawn: Therefore *†* as AC is to BD, so is EG to LM; but by supposition as AC is to BD; so is EG to FH: Therefore *b* LM is equal to FH. But the said LM is *c* parallel to FH: Therefore *d* FL is also parallel to EG; and therefore the angle FLE *e* is equal to the angle LEG. But the angle C is also equal to the said angle LEG, and the angle FLE to the angle FGE *f*: Therefore also the angle C is equal to the angle FGE. But by supposition the angle ABC is equal to the angle EFG: Therefore the third angle BAC is equal to the third angle FEG: Wherefore the triangle ABC is equiangled to the triangle EFG.

Scholium.

g 4. 6.

h 4. 6.

† Now that as AC is to BD, so EG is to LM, it is by some thus demonstrated. Forasmuch as the angle C is equal to the angle GEL, and the angle BDC to the angle LME, each being a right angle, the other angle CBD is equal to the other angle ELM: Therefore *g* as EM is to ML, so is CD to DB. Again, seeing the angle ABC is equal to the angle ELG, and the angle CBD to the angle ELM, the remaining angle ABD is equal to the remaining angle MLG; but the angle ADB is also equal to the angle LMG; and therefore the third angle A is equal to the third angle LGM: Therefore *h* as AD is to DB, so is GM to ML. But it hath been demonstrated, that as

CD

*CD is to DB, so is EM to ML: Therefore as i 14. 5.
AC is to BD, so is EG to LM.*

PROP. LXXX.

If a triangle ABC hath one angle A given, and that the rectangle contained under the sides AB and AC, comprising the given angle A, hath a given reason to the square of the other side BC, the triangle ABC is given by kind.

Constr. For from the points A and B, let there be drawn the perpendiculars AD and BE.

Demonstr. Forasmuch as the angle BAE is given, and also the angle AEB, the triangle ABE is given by a kind; and therefore the reason of AB to BE is given: Therefore the reason of the rectangle of AB and AC to the rectangle of BE and AC is also given (for it is the same reason

a 40. prop.

as of AB to BE.) But the rectangle of AC and BE is equal to the rectangle of BC and AD; for that each of those rectangles is double to the triangle ABC. Therefore the reason of the rectangle of AB and AC to the rectangle of BC and AD is

b 1. 6.

also given. But the reason of the rectangle of AB and AC to the square of BC is given:

Therefore also the reason of the rectangle of BC and AD to the square of BC is given;

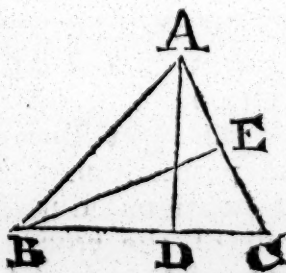
c 41. 1.

and therefore the reason of the right line BC to the right line AD is given. (For that the

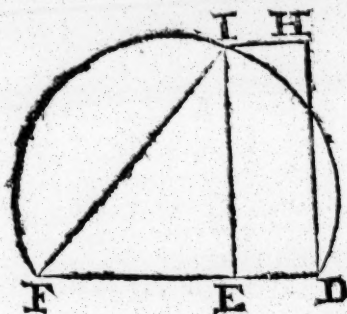
e 1. 6.

rectangle is to the square as AD to BC.) Now let the right line FG, given by position and magnitude, be exposed; and thereon let there be described the segment of a circle FIG;

capable



capable of an angle equal to the angle A .
 And seeing the said angle A is given, also the
 angle in the segment FLG shall be given;
 and therefore *f* the same segment is given by
 position. From the point G let there be erect-
 ed at right angles on the line FG , the line
 GH , which *g* is given by position: Let



h 2. *prop.*

But it is also given by position,
 and the point G is given: Therefore the point
i 27. *prop.* H is *i* also given. Now by the point H let
 there be drawn HI , parallel to FG , and that
k 28. *prop.* line HI shall be given by *k* position. But the
 segment of the circle FIG is also given by
l 25. *prop.* position. Therefore *l* the point I is given.
 Let the right lines IF and IG be drawn, and
 the perpendicular IK : Therefore IK is given
 by position. But the point I is given, as also
m 26. *prop.* each of the points F and G : Therefore *m*
 each of the lines FG , FI , and IG is given
n 39. *prop.* by position and magnitude: Wherefore *n* the
 triangle FIG is given by kind; and seeing
 that as BC is to AE , so is FG to GH , and
o 34. *prop.* so that to GH , IK is equal, as BC is to AE ,
 so is FG to IK , and the angle A is equal to
p 79. *prop.* the angle FIG : Therefore *p* the triangle ABC
 is equiangular to the triangle FIG . But FIG
 is given by kind: Therefore also the triangle
 ABC is given by kind.

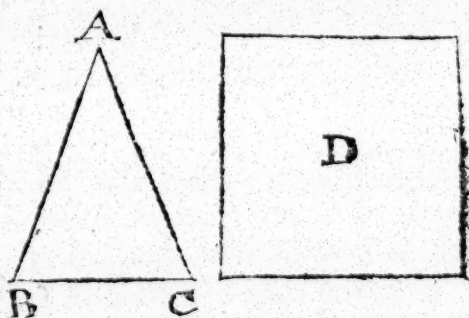
OTHER.

OTHERWISE.

Constr. Let the triangle ABC, whose angle A is given, and the reason of the rectangle contained under AB and AC, to the square of BC be given: I say that the triangle ABC is given by kind.

Demonstr. For seeing the angle A is given, that space of which the square of the line compounded of BAC is greater than the square of BC, *q* hath a given reason to the triangle ABC. *q 67. prop.* Now let that space be D: Therefore the reason of D to the triangle ABC is given. But the reason of the triangle ABC to the rectangle of AB and AC is given; *r* seeing *r 66. prop.* the angle A

is given: Therefore the reason of the space D to the rectangle of AB and AC is given. But the reason of the rectangle of AB and AC



s 8. prop.

to the square of BC is also given: Therefore the reason of the space D to the square of BC is given. Wherefore by compounding, *t* the reason of the space D, with the square of BC to the said square of BC is given: Therefore the reason of the square of the line compounded of BAC, to the square of BC is given; (for that the space D with the square of BC is equal to the square of the line compounded of BAC;) and therefore *u* the reason of the said line compounded of BAC to BC is given. But the angle A is also given: There-

t 6. prop.

u sch. 52. prop.

x 46. prop. Therefore *x* the triangle ABC is given by kind.

PROP. LXXXI.

<u>A</u>	<u>D</u>
<u>B</u>	<u>E</u>
<u>C</u>	<u>F</u>

If of three right lines A, B, and C, proportional to three other proportional right lines D, E, and F; the extremes A and D, C and F, are in a given reason (to wit, as A to D, and C to F,) also the means, B and E shall be in a given reason, and if one extreme hath a given reason to an extreme, and the mean to the mean, the other will have also a given reason to the other.

Demonstr. Forasmuch as the reason of A to D, and of C to F is given, the rectangle of A and D *a* shall have a given reason to the rectangle of C and F. But the rectangle of A and D is equal *b* to the square of B; and the rectangle of C and F to the square of E. Therefore the reason of the square of B to the square of E is given; and therefore *c* the reason of the line B to the line E is also given.

Again, let the reason of A to D, and B to E, be given: I say that the reason of C to F is also given. For seeing that the reason of A to D, and of B to E is given, also the reason of the square of B *d* to the square of E is given. But the square of B is equal to the rectangle of A and C, and the square of E to the rectangle of D and F: Therefore the reason of the rectangle of A and C to the rectangle of D and F is given. But the reason of a side A to a side D is given: Therefore *e* the reason of the other side C to the other side F is also given.

PROP.

P R O P. LXXXII.

If there be four right lines *A, B, C,* *A* _____
and *D,* proportional, as the first *A,* *B* _____
shall be to that line to which the se- *C* _____
cond *B* hath a given reason, so the *D* _____
third *C* shall be to that to which *E* _____
the fourth *D* hath a given reason. *F* _____

Constr. Let *E* be the line to which *B* hath a given reason; and let it be so as that *B* may be to *E,* as *D* is to *F.*

Demonstr. Now the reason of *B* to *E* is given, therefore also the reason of *D* to *F* is given. And seeing that as *A* is to *B,* so is *C* to *D.* And again, as *B* is to *E,* so is *D* to *F,* by reason of equality as *A* is to *E,* so is *C* to *F.* But *E* is that line to which *B* hath a given reason, and *F* that to which *D* also hath a given reason: Therefore as *A* is to that to which *B* hath a given reason, so *C* is to that to which *D* hath a given reason.

P R O P. LXXXIII.

If four right lines *A, B, C,* *A* _____
and *D,* are in such sort to one *B* _____
another, that of any three of them *C* _____
A, B, and *C,* and a fourth *E,* *D* _____
taken proportional, to which that *E* _____
line *D,* which remains of the four *D* _____
lines, hath a given reason, the *E* _____
four lines *A, B, C,* and *E,* are *E* _____
proportional; as the fourth *D* is *E* _____
to the third *C,* so the second *B* *E* _____
shall be to that to which the first *A* hath a given reason.

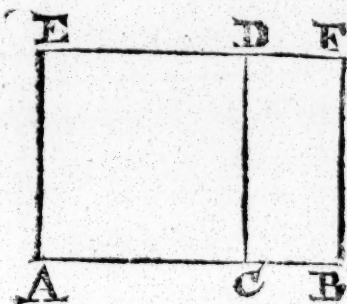
Demonstr. Forasmuch as *A* is to *B,* as *C* is to *E,* the rectangle contained under *A* and *E* is a 16. 6. equal to the rectangle contained under *B* and *C;*

b r. 6.

c 74. prop.

C; and seeing that the reason of D to E is given, also shall be given the reason of the rectangle of A and D to the rectangle of A and E (for *b* it is the same reason as of D to E.) But the rectangle of A and E is equal to the rectangle of B and C: Therefore the reason of the rectangle of A and D to the rectangle of B and C is given. Wherefore *c* as D is to C, so is B to that to which A hath a given reason.

P R O P. LXXXIV.



If two right lines AB and AE comprehending a given space AF in a given angle BAE, and that the one AB be greater than the other AE by a given line CB, also each of the lines AB and AE is given.

a sch. 61.
prop.

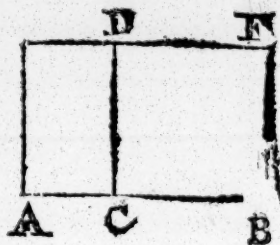
b 59. prop.

Demonstr. For seeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finish the parallelogram AD. Therefore seeing that AE is equal to AC, the reason of AE to AC is given, and the angle A is also given: Therefore *a* AD is given by kind. Wherefore the given space AF is applied to the given right line CB, exceeding it by the given figure AD given by kind; and therefore *b* the breadth of the excess is given. Therefore AC is given. But CB is also given: Therefore the whole AB is given. But AE is also given: Therefore each of the right lines AB and AE is given.

P R O P.

PROP. LXXXV.

If two right lines AC and CD, do comprehend a given space AD in a given angle ACD, the line compounded of those lines AC and CD is given, also each of those lines AC and CD is given.



Constr. For let AC be prolonged to the point B and let CB be put equal to CD, then by the point B let BF be drawn parallel to CD, and so finish the parallelogram CF.

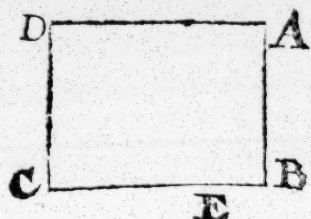
Demonstr. Seeing then that CB is equal to CD, and the angle DCB is given; for that angle that follows is the given angle; and therefore *a* the parallelogram DB is given by kind; and again, seeing that the line compounded of ACD is given, and CB is equal to CD, also AB is given. And thus to the right line AB there is applied the given space AD, deficient by the figure DB given by kind; and therefore *b* the breadths of the defects are also given: Therefore the right lines DC and CB are given. But the compounded line ACD is also given: Therefore *c* each of the lines AC and CD is given.

a sch. 61. prop.

b 58. prop.

c 4. prop.

PROP. LXXXVI.



If two right lines AB and BC , do comprehend a given space AC , in a given angle ABC , the square of the one BC , is greater than the square of the other AB , by a given space (yet in a given rea-

son,) also each of those lines AB and BC shall be given.

- Demonstr.* For seeing that the square of BC is greater than the square of AB by a given space (yet in a certain reason:) Let the given space be taken away, that is to say, the rectangle contained under CB and BE : Therefore
- a 11. def.** a the reason of the remainder, b which is the
 - b 2. 2.** rectangle contained under BC and CE to the square of AB is given. And forasmuch as the rectangle $\dagger$ under AB and BC is given, and
 - c 1. prop.** also that of CB and BE , their c reason is given. But as the rectangle under AB and BC is to the rectangle under CB and EB , d so AB is to BE ; and therefore the reason of AB to BE
 - e 50. prop.** is given: Wherefore e the reason of the square of AB to the square of BE is also given. But the reason of the square of AB to the rectangle under BC and CE is given: Therefore f
 - f 8. prop.** also the reason of the rectangle under BC and CE to the square of BE is given. Wherefore the reason of four times the rectangle under BC and CE to the square of BE is given;
 - g 8. prop.** and by compounding, g the reason of four times the rectangle under BC and CE , with the square of BE to the square of BE is given. But four times the rectangle of BC and CE , with the square of BE , h is the square of the compound line BCE : Therefore the reason of the
 - h 8. 2.** square

square of the compound line BCE to the square of BE is given: Wherefore *i* the reason of the line compounded of BC and CE to BE is given, and by compounding, *k* the reason of the compound of the lines BC, CE, and BE, that is to say, the double of BC to BE is given; and therefore the reason of the only line BC to BE is also given. But as BC is to BE, *l* so the rectangle under BC and BE is to the square of BE: Therefore the reason of the rectangle under BC and BE to the square of BE is given. But the rectangle of BC and BE is given: Therefore *m* the square of BE is also given, and consequently the line BE is given. Wherefore BC is also given, seeing that the reason of BE to BC is given. But the space AC is given, and also the angle B: Therefore *n* AB is given. Wherefore each of the lines AB and BC is given. *i* 51. prop. *k* 6. prop. *l* 11. 6. *m* 2. prop. *n* 57. prop.

Scholium.

† Instead of saying in this place [what is under, &c.] we have used this Word Rectangle, it being manifest by what follows that such was the intention of EUCLIDE, seeing he makes use in the said Demonstration of the second and eighth Proposition of the twelfth Element; and also that the space or Parallelogram given being not rectangled, it may be reduced thereto, making on BC, and in the given point B, a right angle CBA, so as that there will be two Parallelograms constituted on one and the same base EC, and between the same parallels, as in the 69th Proposition by means whereof this conclusion is drawn.

Note, This serves also for the next Prop.

PROP. LXXXVII.



If two right lines AB and BC , do comprehend a given space AC , in a given angle B , the square of the one BC is greater than the square of the other AB , by a given space;

also each of those lines AB and BC shall be given.

Demonstr. For seeing that the square of BC is greater than the square of AB by a given space: Let the given space be taken away, and let the rectangle be contained under BC and BE : Therefore the remainder, *a* which is the rectangle of BC and CE , is equal to the square of AB . And seeing that the rectangle of BC and BE is given, and also the space or rectangle AC , the reason of the said rectangle of BC and BE to AC is given. But as *b* the rectangle of BC and BE is to the rectangle of AB and BC , so is BE to AB : Therefore the reason of BE to AB is given, and therefore *c* the reason of the square of the said DE to the square of AB is also given. But to that square of AB the rectangle of BC and CE is equal: Therefore the reason of the said rectangle of BC and CE to the square of BE is given; and therefore the reason of the quadruple of the said rectangle of BC and CE to the square of BE is also given; and by compounding, *d* the reason of four times the rectangle of BC and CE , with the square of BE , to the said square of BE is given. But four times the rectangle of BC and CE , with the square of BE , *e* is the square of the compound line BCE : Therefore the reason of the square of that compound line BCE to the square of BE

a 2. 2.

b 1. 6.

c 50. prop.

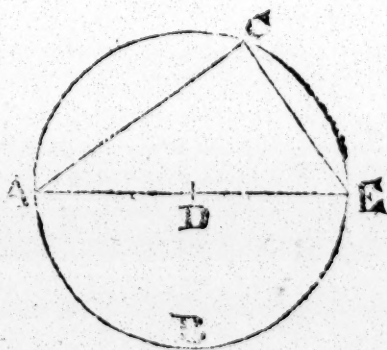
d 6. prop.

e 8. 2.

is also given; and therefore the reason *f* of the *f* 54. *prop.*
compound line BCE to BE is given. Where
fore by compounding *g* the reason of the said *g* 6. *prop.*
compound line BCE and EB, that is to say,
twice BC to BE is also given; therefore the
reason of the only line BC to BE is given.
But the reason of the same BE to AB is also
given: Therefore *h* the reason of AB to BC is *h* 8. *prop.*
given. And seeing that the reason of BC to
BE is given, and that as the said BC is to BE,
so the square of BC *i* to the rectangle of BC *i* 1. 6.
and BE, the reason of the square of BC to the
rectangle of BC and BE is also given. But
the said rectangle of BC and BE is given, it
being that which was taken away, and which
was given. Therefore the square of BC *k* is *k* 2. *prop.*
given, and therefore the line BC is given.
But the reason of the same BC to BA is given,
therefore AB is also given.

P R O P. LXXXVIII.

If in a circle
ABC, given by mag-
nitude, there be
drawn a right line
AC, which shall take
away a segment
ABC, which doth
comprehend a given
angle AEC, that
line AC is given by
magnitude.



Constr. For let D be the center of the circle;
and let the diameter thereof ADE be drawn,
and let EC be joined.

Demonstr. Forasmuch as the angle ACE is
given, for *a* it is a right angle. But the angle *a* 31. 3.
AEC is also given, and therefore the other
angle CAE is given. Wherefore the triangle

G g 3

ACE

b 40. *prop.* ACE *b* is given by kind; and therefore the reason of EA to AC is given. But AE is given by magnitude, seeing that the circle ABC is given by magnitude. Therefore *c* AC is also given by magnitude.

P R O P. LXXXIX.

If in a circle ABC, given by magnitude, there be drawn a right line AC, given by magnitude, the line AC will take away a segment ABC, comprehending a given angle.

Constr. For having taken the point D for the center of the circle, let the diameter ADE be drawn, as also the right line EC.

Demonstr. Forasmuch as each of the right lines AE and AC are given, the reason of the line AE to AC *a* is given; and the angle ACE is a right angle: Therefore *b* the triangle ACE is given by kind, and therefore the angles AEC is given.

P R O P. XC.

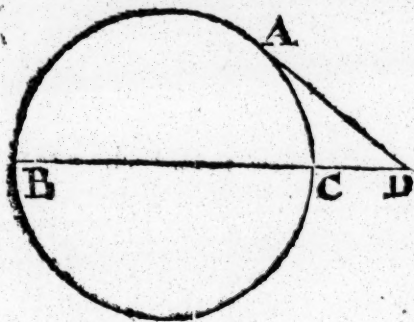
If in the circumference of a circle ABC, given by position and by magnitude, there be taken a given point B, and that from the point B to the circumference of the circle ABC, there doth bend a right line BAC, making a given angle BAC, the other extremity C of the bent line shall be given.

Constr. For let the center of the circle be D, and let the right lines BD and BC be drawn.

Demonstr.

c 6. def. DAC c is given by position, and also the circle
 d 25. prop. ABE, d the point A is given. But the point
 e 26. prop. C is also given: Therefore e the right line
 AC is given by position and by magnitude.

PROP. XCII.



If without a circle ABC, given by position, there be taken some point D, and from that given point there be drawn a right line DB, cutting the circle, the rectangle comprised un-

der the whole line BD, and the part DC, between the point D, and the circumference convex AC, shall be given.

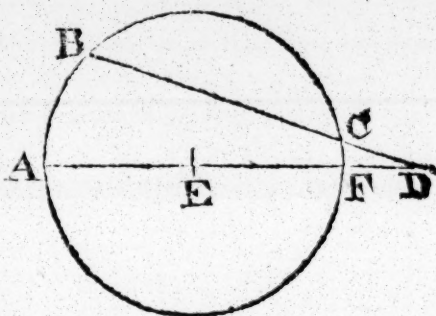
Constr. For from the point D let the right line DA be drawn, which shall touch the circle in the point A.

e 91. prop. Demonstr. Therefore DA a is given by position and magnitude; and therefore the square of the said DA is b given. But the said square of DA is equal c to the rectangle of BD and DC: Therefore the said rectangle of BD and DC is also given.

OTHER.

OTHERWISE.

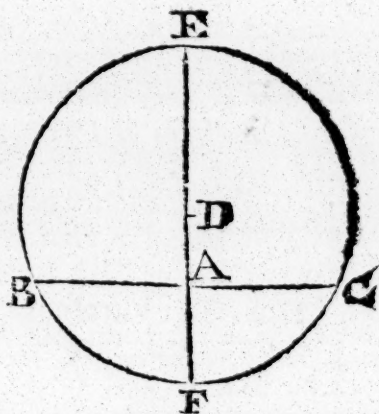
Constr. Let E be the center of the circle, and by the same center let there be drawn from the point D the right line DA.



Demonstr. Forasmuch as each point D and E is given, the right line DE is *d* given by position and by magnitude. But the circle ABC is given by position and by magnitude: Therefore each point A and F *e* is given, and the point D is also given; and therefore *d* each line AD and FD is given. Wherefore the rectangle of the lines AD and DF is also given. But the said rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given. d 26. prop.
e 25. prop.

P R O P. XCIII.

If in a circle given by position there be taken a given point A, and by that point A there be drawn a right line BC to the circle, the rectangle comprised under the segments of the same line BC shall be given.



Constr. For let D be taken for the center of the circle, and having

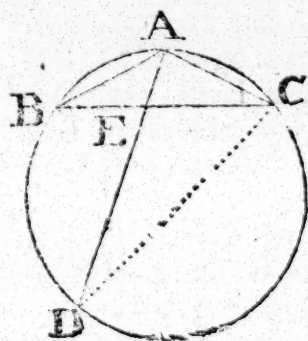
having drawn the right line AD prolong it to the points E and F.

Demonstr. Forasmuch as each point A and D is given, the right line AD *a* is given by position. But the circle BEC is also given by position: Therefore each point E and F is also given by position, and the point A is given. Wherefore each line *b* AE and AF is given: Therefore the rectangle of the same lines AE and AF is given, and is equal to the rectangle *b* of AB and AC: Therefore the said rectangle of AB and AC is given.

a 26. prop.

b 35. 3.

P R O P. XCIV.



If in a circle ABC, given by magnitude, there be drawn a right line BC, which doth take away a segment which doth comprehend a given angle ABC, and that the said angle being in the segment is cut into two equal parts, the line compounded of the right lines BA and

AC, which comprehend the given angle BAC shall have a given reason to the line AD, which doth divide that angle into two equal parts; and the rectangle contained under the line compounded of those lines BA and AC, comprehending the given angle BAC, and that part ED of the intersecting line which is below the segment between the base BC and the circumference, shall be given.

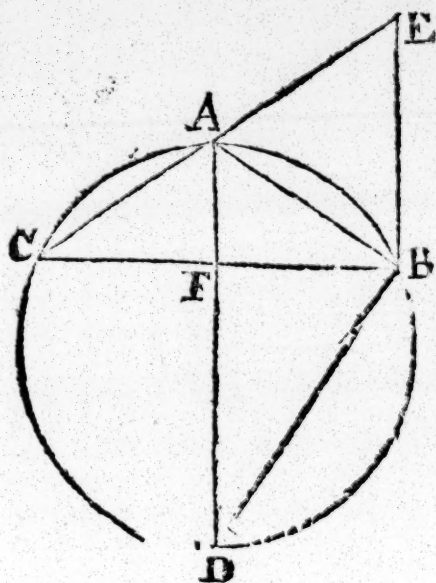
Constr. Let BD be drawn.

Demonstr. Forasmuch as in the circle ABC given by magnitude, there is drawn the right line BC, which takes away the segment BAC comprehending the given angle BAC, that
line

line BC *a* is given; and therefore BD is also *a* 88. *prop.*
 given: Therefore the reason of BC to BD *b* is *b* 1. *prop.*
 given. And seeing that the given angle BAC
 is cut in two equal parts by the right line AD,
 as *c* BA is to CA, so is BE to CE; and by com- *c* 3. 6.
 pounding, as BAC is to CA, so is BC to CE;
 and by permutation, as BAC is to BC, so is
 CA to CE. And seeing that the angle BAE
 is equal to the angle CAE, and the angle
 ACE *d* to the angle BDE, the other angle *d* 21. 3.
 AEC is equal to the other angle ABD; and
 therefore the triangle ACE is equiangled
 to the triangle ABD: Therefore *e* as AC is to *e* 4. 6.
 CE, so is AD to BD. But as AC is to CE,
 so the line compounded of BA and AC is to
 BC: Therefore as the compound line BAC is
 to BC, so is AD to BD; and by permutation,
 as the compound line BAC is to AD, so is
 BC to BD. But the reason of BC to BD is
 given: Therefore the reason of the compound
 line BAC to AD is also given. Moreover,
 I say that the rectangle under the compound
 line BAC and ED is given. For seeing that
 the triangle AEC is equiangled to the trian-
 gle BDE, (for the angle ACE *d* is equal to the
 angle BDE, and the angle AEC *f* to the angle *f* 15. 1,
 BED) as BD is to DE, so is AC to CE. But
 as AC is to CE, so is also the compound
 line BAC to BC: Therefore as the compound
 line BAC is to BC, so is BD to DE. Where-
 fore the rectangle of the compound line BAC
 and DE *g* is equal to the rectangle of BC and *g* 16. 6,
 BD. But the rectangle of BC and BD is
 given, (for that those lines BC and BD are
 given:) Therefore the rectangle under the com-
 pound line BAC and ED is also given.

OTHER.

OTHERWISE.



Constr. Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

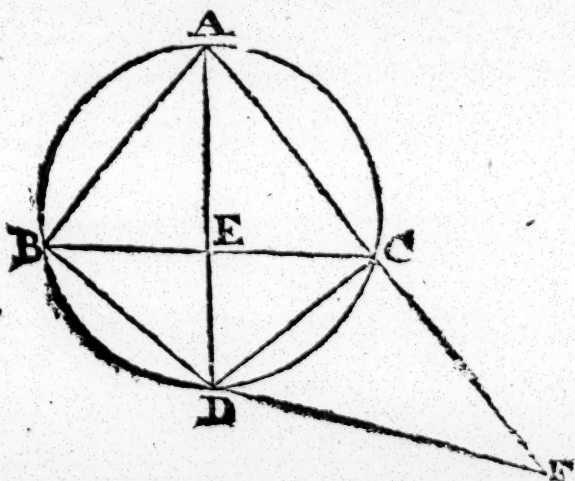
Demonstr. Forasmuch as the angle EAC is double to each of the angles CAD and AEB (for the angle BAC is cut into two equal parts by the line AD, and equal *h*

h 32. 1.
i 5. 1.
k 21. 3.

to the two angles ABE and AEB, which *i* are equal) the angle ABE is equal to the angle CAD, that is to say, *k* to the angle CBD; adding therefore the common angle ABC, the whole angle ABD shall be equal to the whole angle FBE. But the angle ACB is *k* equal to the angle ADB: Therefore the third angle AEB is equal to the third angle BAD; and therefore the triangle CEB is equiangular to the triangle ABD: Wherefore as CE is to CB, so is AD to BD. But the right line CE is compounded of the two lines CA and AB: Therefore as the compound line BAC is to CB, so is AD to BD; and by permutation, as the compound line BAC is to AD, so is CB to BD. But the reason of CB to BD is given, seeing that each of those lines is given: Therefore the reason of the compound line BAC to AD is also given. And seeing that the

the triangle CEB is equiangular to the triangle FBD (for the angle AFC is equal l to the angle BFD, and the angle ECB m to the angle BFD. 12. 3. ADB) as EC is to CB, so is BD to DF. But EC is equal to the compound line BAC: Therefore as the compound line BAC is to CB so is BD to DF. Wherefore n the rectangle of the compound line BAC and DF is equal to the rectangle of CB and BD. But the rectangle of CB and BD is given, considering that each of the lines CB and BD is given: Therefore the rectangle of the compound line BAC and DF is given. 16. 6.

OTHERWISE.



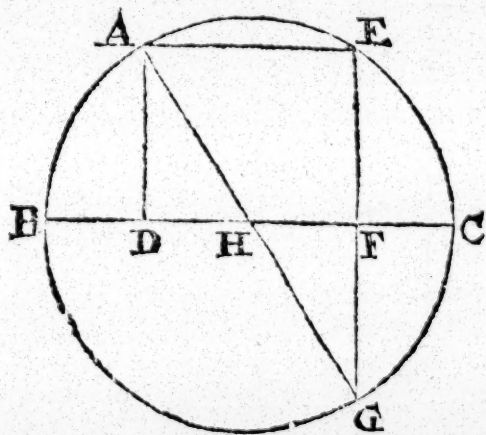
Constr. Let AC be prolonged to F, and let CF be put equal to AB, and let the right lines BD and DF be drawn.

Demonstr. Forasmuch as BA is equal to CF, and o BD to DC, the two sides AB and BD are equal to the two sides CD and DF, each to his corresponding side, and the angle ABD is equal to the angle DCF, p seeing that the four 26, 29. 3. p 22. 3.

94. r.

four sided figure ABDC is within the circle: Therefore the base AD is γ equal to the base DF, and the angle DAB to the angle DFC. But the angle BAD is given, being the half of the given angle BAC) therefore the angle DFC is so also. But DAF is also given: Therefore the triangle ADF is given by kind. Wherefore the reason of FA to AD is given. But AF is the compound of BA and AC, for that CF is equal to AB: Therefore the reason of the compound line BAC to AD is given: The same Demonstration will serve to shew that the rectangle contained under the compound line BAC and ED is given also.

P R O P. XCV.



If in the diameter BC of a circle ABC given by position, there be taken a given point D, and that from that point D there be drawn a right line DA, to the circumference

of the circle. But from the section of the said line there be drawn a right line AE, perpendicular thereto, and by the point E where that perpendicular doth meet with the circumference, there be drawn a parallel EF, to the first line drawn AD, that point F in which the parallel meets with the diameter, is given; and the rectangle contained under the parallel lines AD and EF is also given.

Constr.

Constr. Let the right line EF be prolonged to the point G, and let the right line AG be drawn.

Demonstr. Forasmuch as the angle AEG is a right angle, the right line AG is the diameter of the circle. But BC is also the diameter: Therefore the point H is the center of the circle. Now the point D is given; and therefore *a* the line DH is given by magnitude. But seeing that AD is parallel to EG, and AH equal to GH; *b* DH is equal to FH, and AD to FG: (for the angles AHD and FHG *c* are equal, and DAH and FGH *d* are also equal.) But the line DH is given: Therefore FH is also given. But each of those lines DH and HF is also given by position, and the point H is given: Therefore *e* the point F is also given. And seeing that in the circle ABC given by position, is taken the given point F, and through the same is drawn the right line EFG; the rectangle under EF and FG *f* is given. But FG is equal to AD. Therefore the rectangle comprehended under AD and EF is given. Which was to be demonstrated.

a 26. prop.

b 26. prop.

c 15. 1.

d 26. 1.

e 27. prop.

f 93. prop.

The End of EUCLIDE'S DATA.





A
BRIEF TREATISE
(Added by *FLUSSAS*)
O F
Regular Solids.

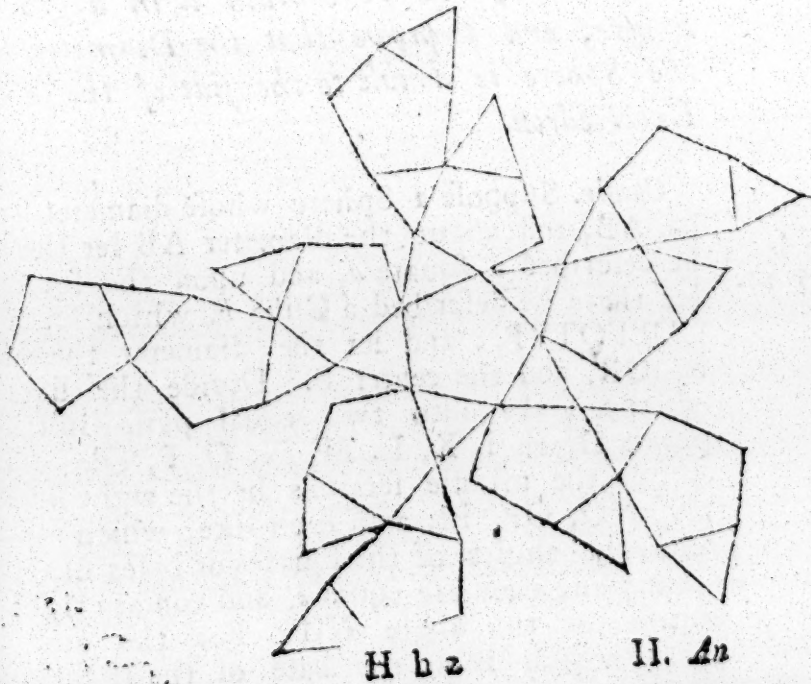
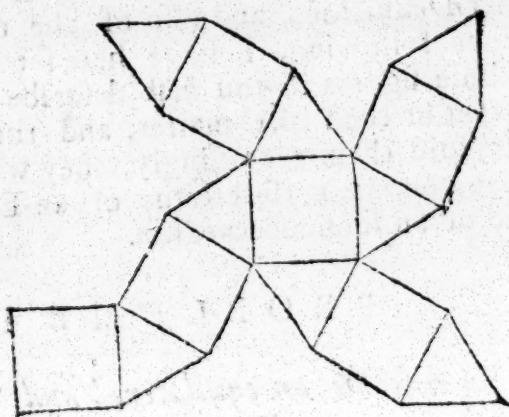
 Regular Solids are said to be composed and mix'd when each of them is transformed into other Solids, keeping still the form, number and inclination of the bases, which they before had to one another; some of which yet are transformed into mix'd Solids, and other some into simple. Into mix'd, as a
H h Dode-

Dodecaedron and an Icofaedron, which are transformed or altered, if you divide their sides into two equal parts, and take away the solid angles subtended of plane superficial figures, made by the lines coupling those middle sections; for the Solid remaining after the taking away of those solid angles, is called an Icofidodecaedron. If you divide the sides of a Cube and of an Octoedron into two equal parts, and couple the sections, the solid angles subtended of the plane superficies made by the coupling lines, being taken away, there shall be left a Solid, which is called an Exoctoedron. So that both of a Dodecaedron and also of an Icofaedron, the Solid which is made shall be called an Icofidodecaedron; and likewise the Solid made of a Cube, and also of an Octoedron, shall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a simple Solid; for if you divide into two equal parts each of the sides of the pyramis, triangles described of the lines which couple the sections, and subtending and taking away the solid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the bases, of all which triangles is produced an Octoedron, to wit, a simple, and not a composed Solid. For the Octoedron hath four bases, like in number, form, and mutual Inclination with the bases of the pyramis, and hath the other four bases with like situation opposite and parallel to the former. Wherefore the application of the pyramis taken twice, maketh a simple Octoedron, as the other Solids make a mix'd compound Solid.

DEFI.

DEFINITIONS.

- I. *An Exoctoedron is a solid figure contained of six equal squares, and eight equilateral and equal triangles.*



H b z

II. An

II. *An Icosidodecaedron is a solid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.*

For the better Understanding of the two former Definitions, and also of the two Propositions following, I have here set two figures, whose figures if you first describe upon pasted Paper or such like matter, and then cut them and fold them accordingly, they will represent unto you the perfect forms of an Exoctaedron, and of an Icosidodecaedron.

PROBLEME I.

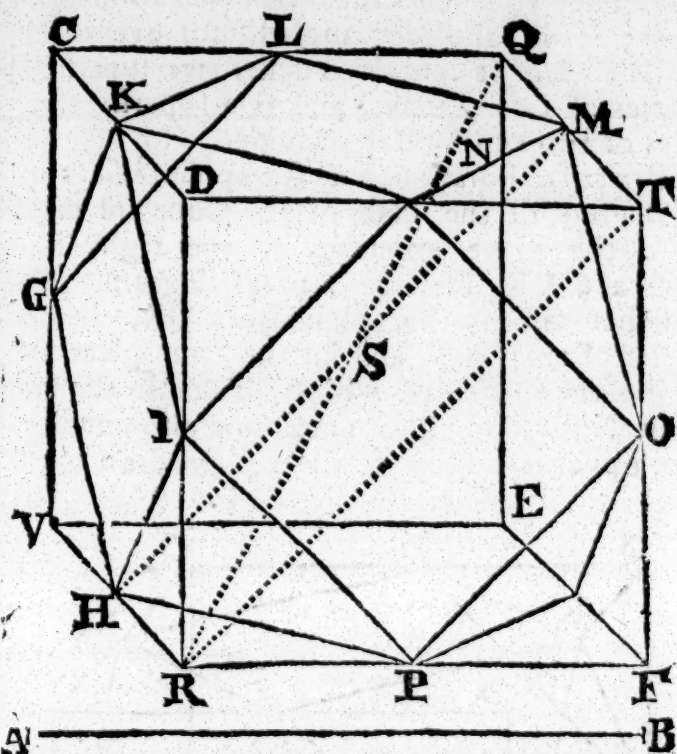
To describe an equilateral and equiangled Exoctaedron, and to contain it in a given Sphere, and to prove that the Diameter of the Sphere is double to the side of the said Exoctaedron.

Constr. Suppose a Sphere whose diameter let be AB, and about the diameter AB let there be described a square *a*, and upon the square let there be described a Cube *b*, which let be CDEFQTVR; and let the diameter thereof be QR, and the center S. Divide the sides of the Cube into two equal parts in the points G, H, I, K, L, M, N, O, P, &c. and couple the middle sections by the right lines IN, NO, OP, PI, and such like, which subtend the angles of the squares or bases of the Cube; and they are equal *c*, and contain right angles, as the angle NIP. For the angle NID, which is at the base of the Isosceles triangle

a 6. 4.
b 15. 13.

c 4. 1.

triangle NDI , is the half of a right angle, and so likewise is the opposite angle RIP . Where-



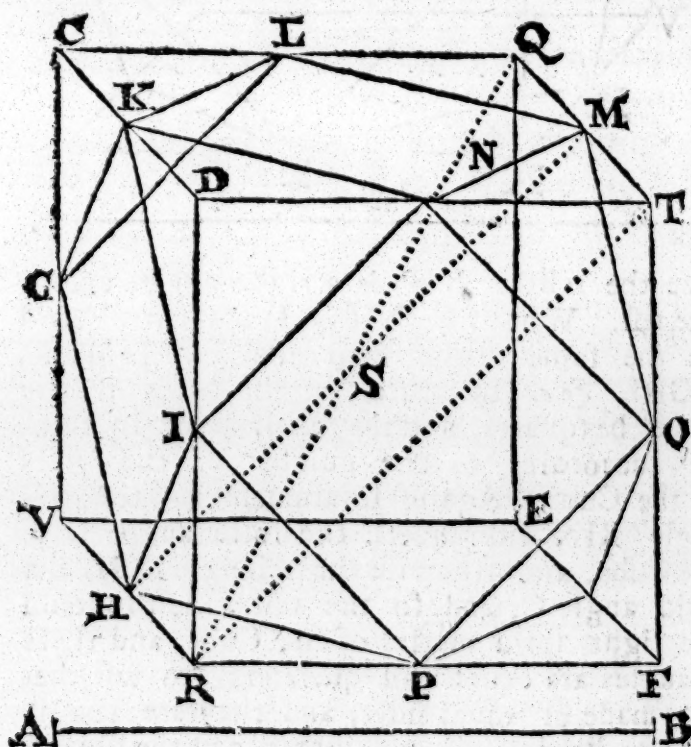
fore the residue NIP is a right angle, and so the rest. Wherefore $NIPO$ is a square. And by the same reason shall the rest $NMLK$, $KGHI$, &c. inscribed in the bases of the Cube, be squares, and they shall be six in number, according to the number of the bases of the Cube. Again, forasmuch as the triangle RIN subtendeth the solid angle D of the Cube, and likewise the triangle KGL the solid angle C , and so the rest which subtend the right solid angles of the Cube, and these triangles are equal and equilateral (to wit) being made of equal sides, and they are the limits or borders of the squares, and the squares the limits or borders of them; as hath been

H h 3

before

before proved. Wherefore LMNOPHGK is an Exoctoedron by the definition, and is equilateral for it is contained of equal subtendant lines; it is also equiangled, for every solid angle thereof is contained under two superficial angles of two squares, and two superficial angles of two equilateral triangles.

Demonstr. Forasmuch as the opposite sides and diameters of the bases of the Cube are parallels, the plane extended by the right lines QT and VR, shall be a parallelogram. And for that also in that plane lyeth QR, the diameter of the Cube, and in the same plane also is the the line MH, which divideth the said plane into two equal parts, and also coupleth the opposite angles of the Exoctoedron; this



line MH therefore divided the diameter into
two

REGULAR SOLIDS.

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two equal parts d ; and also divideth it self d cor. 34. 1. in the same point, which let be S , into two equal parts e . And by the same reason may e 4. 1. we prove that the rest of the lines which couple the opposite angles of the Exoctoedron, do in S the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are set in each of the bases of the Cube. Wherefore making the center the point S , with the distance SH or SM describe a Sphere, and it shall touch every one of the angles equidistant from the point S .

And forasmuch as AB the diameter of the sphere given, is put equal to the diameter of the base of the cube, to wit, to the line RT , and the same line RT is equal to the line MH f , which line MH coupling the opposite angles of the Exoctoedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctoedron. f 33. 1.

Lastly, forasmuch as in the triangle RFT , the line PO doth cut the sides into two equal parts, if it shall cut them proportionally with the bases, to wit, as FR is to EP , so shall RT be to PO g . But FR is double to FP by supposition: Wherefore RT , or the diameter HM , is also double to the line PO , the side of the Exoctoedron. Wherefore we have described, &c. Which was required to be done. g 2. 6.

PROBLEME II.

*To describe an equilateral and equian-
gled Icosidodecaedron, and to comprehend
it in a sphere given, and to prove that the*

H h 4 diame-

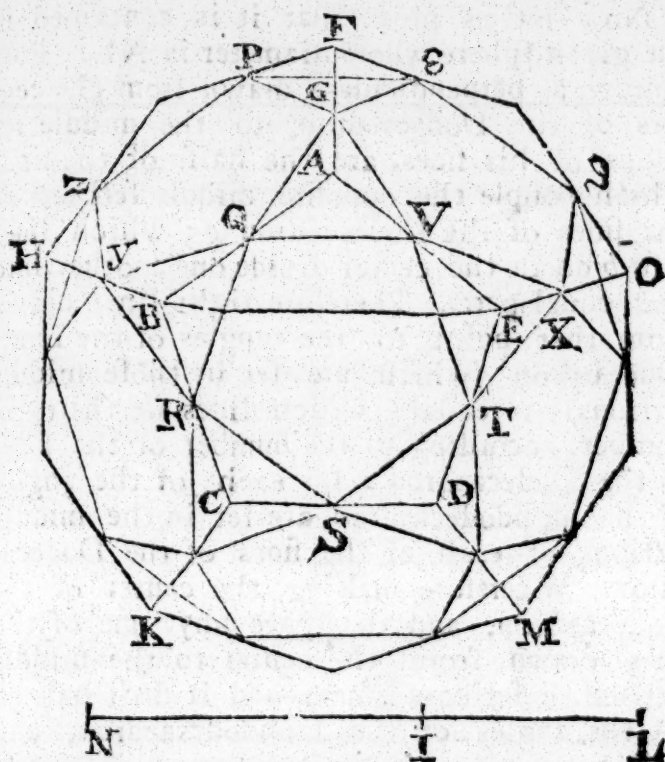
diameter being divided by an extreme and mean proportion, maketh the greater segment double to the side of Icosidodecaedron.

a 30. 6. *Constr.* Suppose that the diameter of the sphere given be NL, *a* divide the line NL, by an extreme and mean proportion in the point I, and the greater segment thereof let be NI; and upon the line NI describe a Cube *b*; and about this Cube let there be circumscribed a Dodecaedron *c*; and let the same be ABCDEFHKMO, and divide each of the sides into two equal parts in the points Q, R, S, T, V, X, Y, Z, P, *e*, *z*, G, &c. and couple the sections with right lines, which shall subtend the angles of the Pentagons, as the lines PG, GV, VQ, QY, YR, RQ, VT, TX, XV, and so the rest.

c 17. 13. *Demonstr.* Forasmuch as these lines subtend equal angles of the Pentagons, and those equal angles are contained of equal sides, to wit, of the halves of the sides of the Pentagons; therefore those subtending lines are equal *d*. Wherefore the triangles GQV, YQR, and VXT, and the rest, which take away solid angles of the Dodecaedron, are equilateral.

d 4. 1. Again, forasmuch as in every Pentagon are described five equal right lines, coupling the middle sections of the sides, as are the lines QV, VT, TS, SR, and RQ, they describe a Pentagon in the plane of the Pentagon of the Dodecaedron. And the said Pentagon is contained in a circle, to wit, whose center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, for that they are perpendiculars upon the bases
cut.

cut *e*. Wherefore the Pentagon QRSTV, is *e* 12. 4.
equiangled *f*. And by the same reason may *f* 11. 4.



the rest of the Pentagons described in the
bases of the Dodecaedron, be proved equal and
like.

Wherefore those Pentagons are twelve in
number: And forasmuch as the equal and like
triangles do subtrend and take away twenty so-
lid angles of the Dodecaedron; therefore the
said triangles shall be twenty in number.
Wherefore we have described an Icosidodeca-
edron by the Definition, which Icosidodeca-
edron is equilateral; for that all the sides of
the triangles are equal and common with the
Pentagons; and it is also equiangled. For
each of the solid angles is made of two su-
perficial

perificial angles of an equilateral Pentagon, and of two superficial angles of an equilateral triangle.

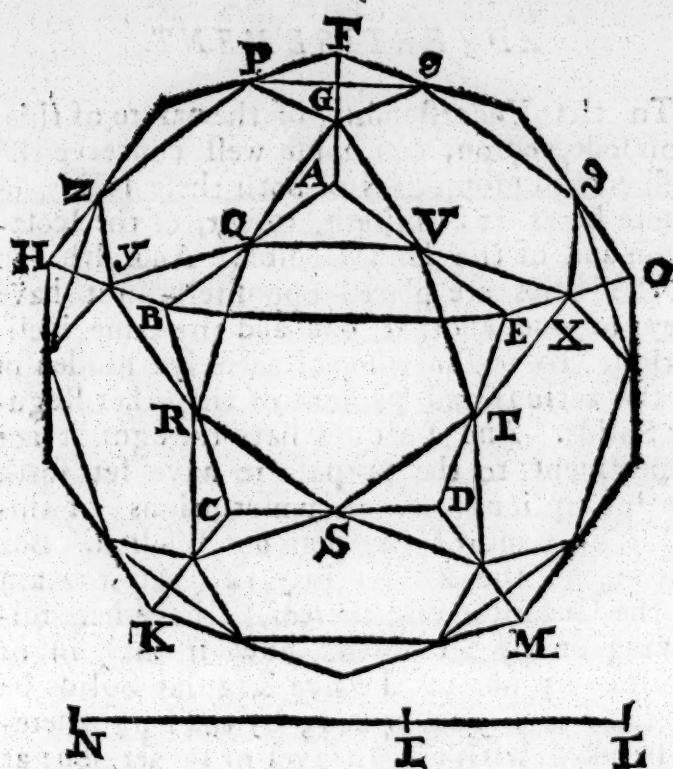
g 3. cor. of
17. 13.
h idem.

Now let us prove that it is contained in the given sphere whose diameter is NL. Forasmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle sections of his sides, are the halves of the lines which couple the opposite middle sections of the sides of the Dodecaedron *g*; which lines also *h* do in the center divide one another into two equal parts. Therefore right lines drawn from that point to the angles of the Icosidodecaedron (which are set in those middle sections) are equal; which lines are thirty in number, according to the number of the sides of the Dodecaedron; for each of the angles of the Icosidodecaedron are set in the middle sections of each of the sides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the space any one of the lines drawn from the center to the middle sections, describe a sphere, and it shall pass by all the angles of the Icosidodecaedron, and shall contain it.

i 4. cor.
17. 13.

And forasmuch as the diameter of this solid, is that right line whose greater segment is the side of the Cube inscribed in the Dodecaedron *i*, which side is NI by supposition. Wherefore that solid is contained in the sphere given, whose diameter is put to be the line NL.

Now



Now let us prove that the great segment of the diameter is double to QV the side of the solid. Forasmuch as the sides of the triangle AEB, are in the points Q and V divided into two equal parts, the lines QV and BE are parallels *k*. Wherefore as AE is to AV, so is EB to VQ *l*. But the line AE is double to the line AV. Wherefore the line BE is double to the line QV *m*. Now the line BE is equal to NI, or to the side of the Cube *n*; which line NI is the greater segment of the diameter NL. Wherefore the greater segment of the diameter given is double to the side of the Icosidodecaedron inscribed in the given sphere. Wherefore we have described, &c. Which was required to be done.

k cor. 39.1.

l 2. 6.

m 4. 6.

n 2 cor. of

17. 13.

ADVER.

ADVERTISEMENT.

To the Understanding of the nature of this Icosidodecaedron, you must well conceive the passions and proprieties of both these solids, of whose bases it consisteth, to wit, of the Icosaedron and of the Dodecaedron. And altho' in it the bases are placed oppositely, yet have they to one another one and the same inclination. By reason whereof there lye hidden in it the actions and passions of the other Regular Solids. And I would have thought it not impertinent to the purpose to have set forth the inscriptions and circumscriptions of this Solid, if want of time had not hindred. But to the end the Reader may the better attain to the Understanding thereof, I have here following briefly set forth, how it may in or about every one of the five Regular Solids be inscribed or circumscribed; by the help whereof he may, with small travel or rather none at all, having well poised and considered the Demonstrations appertaining to the foresaid five Regular Solids, demonstrate both the inscription of the said Solids in it, and the Inscription of it in the said Solids.

*Of the Inscriptions and Circumscriptions of
an Icosidodecaedron.*

An Icosidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which subtend the solid angles of the Dodecaedron, which solid angles are twenty in number, and are placed in the same order in which the solid angles of the Dodecaedron taken away, or subtended by them, are. And by that reason it shall receive a Cube and a
Pyramis

Pyramis contained in the Dodecaedron, when as the angles of the one, are set in the angles of the other.

An Icosidodecaedron receiveth an Octoedron, in the angles cutting the six opposite sections of the Dodecaedron, even as if it were a simple Dodecaedron.

And it containeth an Icosaedron, placing the twelve angles of the Icosaedron in the same centers of the twelve Pentagons.

It may also by the same reason be inscribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular bases concentrical with four bases of the Pyramis, after the same manner that you inscribed an Icosaedron in a Pyramis; so likewise may it be inscribed in an Octoedron, if you make eight bases thereof concentrical with the eight bases of the Octoedron. It shall also be inscribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the bases of the Cube. Again, you shall inscribe it in an Icosaedron, when the triangles compassed in of the Pentagon bases, are concentrical with the triangles which make a solid angle of the Icosaedron.

Lastly, It shall be inscribed in a Dodecaedron, if you place each of the angles thereof in the middle sections of the sides of the Dodecaedron, according to the order of the Construction thereof.

The opposite plain superficies also of this solid are parallels. For the opposite solid angles are subtended of parallel plain superficies, as well in the angles of the Dodecaedron subtended by triangles, as in the angles of the Icosaedron subtended of Pentagons, which thing may easily be demonstrated. Moreover, in this solid are infinite properties and passions, springing of the solids whereof it is composed.

Where-

Wherefore it is manifest, that a Dodecaedron and an Icosaedron mixed, are transformed into one and the self same solid of an Icosidodecaedron. A Cube also and an Octoedron are mixed and altered into another solid, to wit, into one and the same Exoctoedron. But a Pyramis is transformed into a simple and perfect solid, to wit into an Octoedron.

If we will frame these two solids joined together into one solid, this only must we observe.

In the Pentagon of a Dodecaedron inscribe a like Pentagon, and let its angle be set in the middle sections of the pentagon circumscribed, and then upon the said Pentagon inscribed, let there be set a solid angle of an Icosaedron, and so observe the same order in each of the bases of the Dodecaedron, and the solid angles of the Icosaedron, set upon these pentagons shall produce a solid consisting of the whole Dodecaedron, and whole Icosaedron. In like sort, if in every base of the Icosaedron, the sides being divided into two equal parts, be inscribed an equilateral triangle, and upon each of those equilateral triangles be set a solid angle of a Dodecaedron, there shall be produced the same solid consisting of the whole Icosaedron, and of the whole Dodecaedron.

And after the same order, if in the bases of a Cube be inscribed squares subtending the solid angles of an Octoedron, or in the bases of an Octoedron be inscribed equilateral triangles subtending the solid angles of a Cube, there shall be produced a solid consisting of either of the whole Solids, to wit, of the whole cube, and of the whole Octoedron.

But equilateral triangles inscribed in the bases of a Pyramis, having their angles set in the middle sections of the sides of the Pyramis,

and

and the solid angles of a Pyramis, set upon the said equilateral triangles, there shall be produced a solid consisting of two equal and like pyramids.

And now if in these solids thus composed, you take away the solid angle, there shall be restored again the first composed solids, to wit, the solid angles taken away from a Dodecaedron and an Icosaedron composed into one, there shall be left an Icosidodecaedron, the solid angles taken away from a Cube and an Octoedron composed into one solid, there shall be left an Exoctoedron. Moreover, the solid angles taken away from two pyramids composed into one solid, there shall be left an Octoedron.

Of the nature of a trilateral and equilateral Pyramis.

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal squares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whose angles are set in the middle sections of the sides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the sides of the Pyramis, and that which is left is an Octoedron, inscribed in the Pyramis, in which all the solids inscribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the base, is double to the diameter of the Cube inscribed in it.

4. And a right line coupling the middle sections of the opposite sides of the Pyramis is triple to the side of the same Cube.

5. The

*Each of the four
equilateral
triangles
is a cube*

5. The side also of a Pyramis is triple to the diameter of the base of the Cube.

6. Wherefore the same side of the Pyramis is in power double to the right line which coupleth the middle section of the opposite sides.

7. And it is in power sesquialter to the perpendicular which is drawn from the angle to the base.

8. Wherefore the perpendicular is in power sesquitertia to the line which coupleth the middle sections of the opposite sides.

9. A Pyramis and an Octoedron inscribed in it, also an Icosaedron inscribed in the same Octoedron, do contain one and the same sphere.

Of the nature of an Octoedron.

1. Four perpendiculars of an Octoedron, drawn in four bases thereof from two opposite angles of the said Octoedron, and coupled together by those four bases, describe a Rhombus, or Diamond figure; one of whose diameters is in power double to the other diameter.

2. For it hath the same proportion that the diameter of the Octoedron hath to the side of the Octoedron.

3. An Octoedron and an Icosaedron inscribed in it, do contain one and the same sphere.

4. The diameter of the solid of the Octoedron is in power sesquialter to the diameter of the circle which containeth the base, and is in power duple superbi-partiens tertias (that is, as 8 to 3,) to the perpendicular or side of the foresaid Rhombus; and moreover
is

is in length triple to the line which coupleth the centers of the next bases.

5. The angle of the inclination of the bases of the Octoedron, doth, with the angle of the inclination of the bases of the Pyramis, make angles equal to two right angles.

Of the nature of a Cube.

1. The diameter of a Cube is in power sesquialter to the diameter of his base.

2. And is in power triple to his side.

3. And unto the line which coupleth the centers of the next bases, it is in power sextuple.

4. Again, the side of the Cube, is to the side of the Icosaedron inscribed in it, as the whole is to the greater segment.

5. Unto the side of the Dodecaedron, it is as the whole is to the lesser segment.

6. Unto the side of the Octoedron it is in power duple.

7. Unto the side of the Pyramis it is in power subduple.

8. Again, the Cube is triple to the Pyramis, but to the cube the Dodecaedron is in a manner double. Wherefore the same Dodecaedron is in a manner sextuple to the said Pyramis.

Of the nature of the Icosaedron.

1. Five triangles of an Icosaedron, do make a solid angle, the bases of which triangles make a Pentagon. If therefore from the opposite bases of the Icosaedron be taken the other Pentagon by them described, these pentagons shall in such sort cut the diameter of the Icosaedron which coupleth the foresaid opposite angles,

l i

that

that that part which is contained between the planes of these two Pentagons shall be the greater segment, and the residue which is drawn from the plane to the angle, shall be the lesser segment.

2. If the opposite angles of two bases joined together, be coupled by a right line, the greater segment of that right line is the side of the Icosaedron.

3. A line drawn from the center of the Icosaedron to the angles, is in power quintuple to half that line which is taken between the Pentagons, or of the half of that line which is drawn from the center of the circle which containeth the foresaid Pentagon, which two lines are therefore equal.

4. The side of the Icosaedron containeth in power either of them, and also the lesser segment, to wit, the line which falleth from the solid angle to the pentagon.

5. The diameter of the Icosaedron containeth in power the whole line, which coupleth the opposite angles of the bases joined together, and the greater segment thereof, to wit, the side of the Icosaedron.

6. The diameter also is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon composed of the sides of the Icosaedron.

7. The dimetient containeth in power the right line which coupleth the centers of the opposite bases of the Icosaedron, and the diameter of the circle which containeth the base.

8. Again, the said dimetient containeth in power the diameter of the circle which containeth the Pentagon, and also the line which is drawn from the center of the same circle to the circumference; that is, it is quintuple

to

to the line drawn from the center to the circumference.

9. The line which coupleth the centers of the opposite bases, containeth in power the line which coupleth the centers of the next bases, and also the rest of that line of which the side of the Cube inscribed in the Icosaedron is the greater segment.

10. The line which coupleth the middle sections of the opposite sides, is triple to the side of the Dodecaedron inscribed in it.

11. Wherefore, if the side of the Icosaedron and the greater segment thereof be made one line, the third part of the whole is the side of the Dodecaedron inscribed in the Icosaedron.

Of the Dodecaedron.

1. The diameter of a Dodecaedron containeth in power the side of the Dodecaedron, and also that right line to which the side of the Dodecaedron is the lesser segment, and the side of the Cube inscribed in it is the greater segment, which line is that which subtendeth the angle of the inclination of the bases, contained under two perpendiculars of the bases of the Dodecaedron.

2. If there be taken two bases of the Dodecaedron, distant from one another by the length of one of the sides, a right line coupling their centers being divided by an extreme and mean proportion, maketh the greater segment the right line which coupleth the centers of the next bases.

3. If by the centers of five bases set upon one base, be drawn a plain superficies, and by the centers of the bases which are set upon the opposite base, be drawn also a plain superficies, and then be drawn a right line, coupling the centers of the opposite bases,

1 i 2 that

that right line is so cut, that each of his parts set without the plain superficies, is the greater segment of that part which is contained between the planes.

4. The side of the Dodecaedron is the greater segment of the line which subtendeth the angle of the Pentagon.

5. A perpendicular line drawn from the center of the Dodecaedron to one of the bases, is in power quintuple to half the line which is between the planes.

6. And therefore the whole line which coupleth the centers of the opposite bases, is in power quintuple to the whole line which is between the said planes.

7. The line which subtendeth the angle of the base of the Dodecaedron, together with the side of the base, are in power quintuple to the line which is drawn from the center of the circle which containeth the base, to the circumference.

8. A section of a sphere containing three bases of the Dodecaedron, taketh a third part of the diameter of the said sphere.

9. The side of the Dodecaedron and the line which subtendeth the angle of the Pentagon, are equal to the right line which coupleth the middle sections of the opposite sides of the Dodecaedron.



THE
THEOREMS
OF
ARCHIMEDES.

Concerning the Sphere and Cylinder, Investigated by the *Method of Indivisibles*, and briefly Demonstrated by the Reverend and Learned Dr. ISAAC BARROW.

THE main Design of *Archimedes* in his *Treatise of the Sphere and Cylinder*, is to resolve these four Problems.

1. To find the proportion of the superficies of a sphere to any determinate circle; or to find a circle equal to the superficies of a given sphere.

2. To find the proportion of the superficies of any segment of a sphere to any determined circle; or to find a circle equal to the superficies of any assign'd segment.

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3. To

3. To find the proportion of the sphere it self (or of its solid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given sphere.

4. To find the proportion of a segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given segment.

These four Problems *Archimedes* prosecutes separately, and lays down Theorems immediately subservient to their solution; but we reduce them to two: For since an Hemisphere is the segment of a sphere, and the method of finding out its relations, in respect to the superficies and solid content, is comprehended in the general method of investigating the proportion of the segments: And from the superficies and solid content of an Hemisphere already found, the double of them, (that is the superficies and content of the whole sphere) is at the same time given. And indeed 'tis superfluous and foreign from the Laws of good Method, to investigate their relations distinctly and separately; so that if it were not a crime, I might on this account, blame even *Archimedes* himself.

The whole manner therefore is reduc'd to these two Problems.

1. To find the proportion of the superficies of any segment of a sphere, to a determinate circle; or to find a circle equal to the superficies of a given segment.

2. To find the proportion of the solidity of any segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an assign'd segment of a sphere.

I shall resolve these Problems by another much easier and shorter method: In which the order being inverted, first, I shall seek the solidity of a segment, and from thence reduce

its superficies ; a thing which is in my judgment well worth observing, and perform'd (as I know of) by none.

First therefore, for finding the solidity of a segment, I shall lay down two, commonly known and receiv'd Suppositions, viz.

1. That a series of magnitudes proceeding in Arithmetical Progression from nothing (inclusive) or whose common difference is equal to the least magnitude, is subduple of as many quantities equal to the greatest ; (i. e. subduple of the product of the greatest term and number of terms :) So that if the sum of the terms be called z , the greatest term g , and the number of terms n , then will

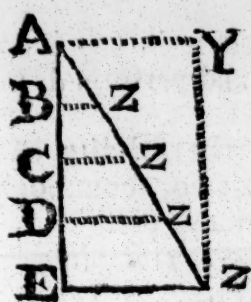
$$z = \frac{ng}{2}, \text{ or } 2z = ng.$$

The truth of this Proposition will easily appear by expressing the series twice, and inverting the order ;

$$\begin{array}{ccccccc} 0, & a, & 2a, & 3a, & 4a. \\ 4a, & 3a, & 2a, & a, & 0. \end{array}$$

For so the difference always being equal to the least quantity, 'twill be evident that each two correspondent terms taken together are equal to the greatest term ; and also, that the series taken twice is equal to the greatest term repeated as many times as there are terms. i. e. the last term drawn into the number of terms.

We have in a triangle a very clear and easy example of this most useful Proposition, which is prov'd hence, to be half a parallelogram having the same altitude, and standing on the same base.

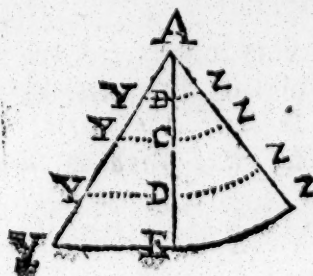


Suppose the altitude AE of the triangle AEZ to be divided into parts indefinitely many and small AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Divisions ; all these proceed from nothing

in an Arithmetical progression, and consequently the sum of 'em all, (that is, the triangle AEZ) is subduple of the greatest EZ drawn into the altitude AE, by which the sum of the terms is express'd, that is, subduple of the Parallelogram EY, whose base is EZ, and altitude AE.

But the illustration of the Rule will conduce more to our design by inferring hence, *That a circle is equal to half of the radius drawn into the circumference*, after this manner. Conceive a circle to consist of as many concentric Peripheries as their are points or equal parts indefinitely many and small in the radius. These Peripheries, as well as their radii, proceed from the center or nothing in an Arithmetical progression ; and therefore their sum, that is, the whole circle is equal to half the greatest (or extreme circumference) drawn into the number of terms, that is, the radius.

After the same manner we may suppose the sector AEZ to consist of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely small) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progression, the sector



also

also will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident also after this manner: Let us suppose the right line EY to be perpendicular to the radius AE, draw the right line AY, and from the points B, C, D, of division in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Because EY : DY ($\because$ rad. AE : rad. AD) $\therefore$ Arc. EZ : Arc DZ, and EY = EZ, then will DY = Arc DZ; and in like manner will CY = CZ, and BY = BZ. whence the triangle AEY will be = to the sector

$$\text{AE} \times \text{EY} \quad \text{AE} \times \text{EZ}$$

AEZ, that is, $\frac{\text{AE} \times \text{EY}}{2} \cdot \left(\frac{\text{AE} \times \text{EZ}}{2} \right) = \text{se-}$

ctor AEZ. By this means we collect that celebrated Theorem of Archimedes, that a circle is equal to a triangle, whose base is equal to the radius, and altitude equal to the periphery of the circle; and that without any inscription or circumscription of figures, by only supposing that the Area or Superficies of the circle consists of infinitely many concentric Peripheries. Which method of Indivisibles, (now first of all known to me) seems no less evident (nay more evident) and perhaps less fallacious than that wherein planes are supposed to consist of parallel right lines, and solids of parallel planes; as hereafter shall be evident, when we shall collect, by this method, the proportions of spheric and cylindric superficies to one another, by knowing the solid content; and on the other hand, the solid content, by knowing the superficies, with admirable facility, and most full satisfaction in those things which are rigidly gathered by pure Geometry.

Let us suppose a series of quantities to proceed from 0 (inclusive) in a duplicate Arithmetic progression, that is, 0, 1, 4, 9, 16, &c. the squares of

of numbers in a simple Arithmetic progression, 0, 1, 2, 3, 4, &c. And the triple of this series will always exceed the greatest term multiplied by the number of terms; but the number of terms increasing the proportion, continually approximates till at last it comes to an equality, when the number of terms is increased in infinitum,

$$3 \times 0 + 1 = 3 \quad \frac{3}{2}$$

$$2 \times 1 = 2. \quad \frac{2}{2}$$

$$3 \times 0 + 1 + 4 = 15. \quad \frac{15}{12} = \frac{5}{4}$$

$$3 \times 4 = 12. \quad \frac{12}{12} = \frac{4}{4}$$

$$3 \times 0 + 1 + 4 + 9 = 42. \quad \frac{42}{36} = \frac{7}{6}$$

$$4 \times 9 = 36. \quad \frac{36}{36} = \frac{6}{6}$$

$$3 \times 0 + 1 + 4 + 9 + 16 = 90. \quad \frac{90}{80} = \frac{9}{8}$$

$$5 \times 16 = 80. \quad \frac{80}{80} = \frac{8}{8}$$

$$3 \times 0 + 1 + 4 + 9 + 16 + 25 = 165. \quad \frac{165}{150} = \frac{11}{10}$$

$$6 \times 25 = 150. \quad \frac{150}{150} = \frac{10}{10}$$

As for example, if the terms be two, the triple of the terms will be to the greatest term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and so continually: So that the antecedents of these proportions always mutually exceed one another by the number 2; and so every antecedent its consequent by 1. Whence it is evident that by how much the greater the number of terms is, by so much the more the proportion tends to equality. So 100 to 99 is less distant from the proportion of equality than 10 to 9. From hence, supposing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the squares of the numbers, 0, 1, 2, 3, 4, &c. will be equal to as many quantities equal to the greatest term,

The

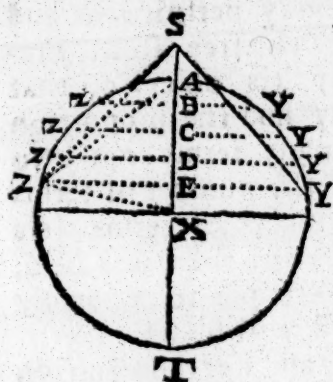
The same, as to the substance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonstrated by our Learned Country-man *Dr. Wallis*: However, I thought fit to illustrate the matter by this method, as being not unworthy our Consideration, and very perspicuous and intelligible in this, that 'tis free from Fractions: And by the way 'tis observ'd, that from hence we may easily find the proportion of a series triple to as many terms equal to the greatest, viz. as twice the number of terms less one, to twice the number of terms less two. So that if the number of the terms be 6, the proportion of a series triple to as many terms equal to the greatest will be as 11.

It will be a very easy and apt Illustration of this Rule, if we infer hence, *That a Cone is subtriple of a Cylinder, having an equal base and altitude.* For let us suppose the altitude *AE* of the cone *ZY* to be divided into equal and indefinitely many parts, by as many parallel right lines *ZY*, and the lines *ZY* will be as the numbers 1, 2, 3, 4, &c. and the squares or circles constituted upon the diameters *ZY*, as 1, 4, 9, 16, &c. whence all those circles, or the whole cone *AZY* (made up of the same) will be subtriple of as many circles equal to the greatest, constituted on the greatest diameter *ZEY*, that is, subtriple of a cylinder whose base is *AEY*, and altitude *AE*.



There occur two other most apt examples of this Rule, viz. by inferring, *That the complement of a Semiparabola is subtriple of a parallelogram having the same base and height*; as also, *That the space comprehended by the Spiral*
and

and Radius is subtriple of the circle in which the spiral is generated : But of these in another place. Wherefore to go on with what we began, these two Rules being supposed ; let us conceive ZAY to be a segment of a



sphere, X its center, AT its diameter, and ZAYT a great circle passing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely many equal parts; and let us imagine parallel lines to be drawn thro' the points of Division, generating cir-

cles in the sphere, whose Radii let be BZ, CZ, DZ, and diameters ZY. I suppose the segment of a sphere to consist of all these parallel circles, whose number is as great as that of the points, or equal indefinitely many small parts in the Axe AE, according to the known *Method of Indivisibles*.

But now for brevity's sake, let the diameter AT be called d , and the radius of the sphere r (if need be,) and the Axe AE, by which the number of terms is express'd, call n , and one of the equal parts a ; which being settled, 'tis evident, (*by the Elements*) that $BZ^2 = AB \times BT = a \times d - a = ad - a^2$, and in like manner $CZ^2 = AC \times CT = 2a \times d - 2a = 2ad - 4a^2$, and by the same reasoning $DZ^2 = AD \times DT = 3ad - 9a^2$, and $EZ^2 = AE \times ET = 4ad - 16a^2$, &c. that is, that the squares of the radii of the circles ZY are to one another as the rectangles $ad, 2ad, 3ad, 4ad$, &c. (which proceed in an Arithmetical Progression from 0) less by the square $a^2, 4a^2, 9a^2$,

$9a^2$, $16a^2$, &c. which go on as the squares of the numbers, 1, 2, 3, 4, &c.) But by our first Rule, all the Rectangles o, ad , $2ad$, $3ad$, $4ad$, &c. are equal to half as many terms equal to the greatest $AE \times AT$ or nd , that is,

$$= \frac{nd \times n}{2} a^2$$

Moreover, by our second Rule, all the squares o, a^2 , $4a^2$, $9a^2$, $16a^2$, &c. taken together, are equal to a third part of as many terms equal to the greatest AE^2 or n^2 , that

$$\text{is, } = \frac{n^2 \times n}{3} a^2$$

Wherefore all the squares described upon the radii BZ, CZ, DZ, EZ, conjunctly, are

equal to the difference $\frac{ndn}{2} - \frac{nnn}{3}$, (or the terms being reduc'd to the same denomination,) $\frac{3ndn - 2nnn}{6}$

and their quadruple, that is, all

the squares described upon the diameter ZY, $\frac{12ndn - 8n^3}{6}$ or $\frac{6ndn - 4n^3}{3}$ are equal to

Whence a segment of a sphere is equal to a Cylinder, the diameter of whose base is the side of a square equal to $6nd - 4n^2$, and altitude is $\frac{3}{4}n$; or to a cone having the same base, but the altitude n . or which is all one, having

a base whose radius is $\sqrt{\frac{6nd - 4n^2}{4}}$ or $\sqrt{\frac{3nd - 2n^2}{2}}$

and altitude n as before. Which Cone we may change into a Cone upon the same base ZY with the segment ZAY, by saying, as ZE^2 (i. e. $dn - n^2$) to $\frac{3}{4}nd - n^2$ or (both

(both terms being divided by n) as $d - n$ to $\frac{2}{3} d - n$, so reciprocally n to the altitude of the Cone sought: Or in the figure by making, as TE to $TE + XA$, so is EA to ES . For ES will be the altitude of the Cone ZSR equal to the segment of the sphere ZAY . Which is a noted Theorem of Archimedes, demonstrated by him with so much labour and prolixity.

Hence, if the given segment be a Hemisphere, and so $n = \frac{1}{2} d$ or r , then d or $2r$ will be the altitude of a Cone, which having a base equal to the base of the Hemisphere, (or to the greatest circle in the sphere,) will be equal to the Hemisphere. And a Cone whose base is double of the greatest circle, and the altitude $2r$, or the Cylinder whose bases is $\frac{3}{2}$ of the greatest circle, and altitude $2r$ will be equal to the whole Sphere. Whence the whole Sphere is $\frac{2}{3}$ of a Cylinder the diameter of whose base is $2r$, and the altitude also $2r$. And this is the chief Theorem of Archimedes, viz. That a sphere subsesquialter or $\frac{2}{3}$ of that Cylinder, whose Altitude and Diameter of the base is equal to the Diameter of the Sphere.

Furthermore, not to pass over any thing in our Author which seems to be to our purpose:

If to the sum first found, representing a seg.

$6ndn - 4nnn,$ $2d^2n - 6dn^2 + 4n^3$
ment, viz. ————— we add —————

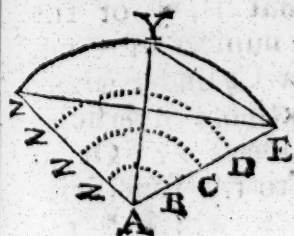
$$\left(= \frac{4dn - 4n^2}{3} \times \frac{d - 2n}{2} = \frac{4}{3} ZE^2 \times XE \right) \text{ repre-}$$

senting the Cone ZXY , the aggregate $\frac{2}{3} ddn$ will represent the Sector of the Sphere $ZX- YA$, which for that reason will be equal to a Cylinder, the diameter of whose base $\sqrt{dn}$, and

and the altitude $\frac{2}{3} d$, or to a Cone, the diameter of whose base is $\sqrt{dn}$, and the altitude $2d$, or also in a Cone, the Radius of whose base is $\sqrt{dn}$, and the altitude $\frac{1}{3} d = r$ (it being reciprocally as $4dn : dn :: 2d : \frac{1}{3} d$,) that is, to a Cone, the Radius of whose base is the Line AZ, drawn from the vertex to the circumference of the base of the segment, (for $AZ^2 = TA \times AE = dn$,) and the altitude r . And this is the next famous Theorem of *Archimedes*, concerning the solidity of the sector of the Sphere, viz. *That the sector of a sphere is equal to a Cone, whose base is a circle described by a Radius equal to a line drawn from the vertex to the circumference of the base of the segment, and whose altitude is equal to the Radius of the sphere.*

And thus I think I have compleated that which belongs to the solidity of a sphere, and its parts, with sufficient brevity and perspicuity. From hence we shall deduce the Resolution of the other Problem, which I proposed concerning the surface of the segment of a sphere; and then of the whole sphere. To obtain this, as we supposed before, a Circle to consist of concentric Peripheries, and the Sector of a Circle, of concentric Arcs, (in the number of which, the greatest, and the least or a point is reckon'd : So now we suppose spheres to consist of concentric spherical superficies, and the Sectors of Spheres, of

like concentric superficies; as for example, the sector of the sphere ZAE, of the superficies BZ, CZ, DZ, EZ, &c.) which supposition indeed seems so easy and natural, that in my judgment 'tis sufficient



cient only to propose it; neither is a further explication wanting to gain an assent to it.

2. We suppose these spherical superficies to be in a duplicate Ratio of the Radius of the spheres: This is the common affection of all like superficies, and it seems to agree very well with the superficies of spheres, because they appear to be most uniform and similar. But this Supposition might easily be evinc'd and establish'd by the same sort of arguing, as spheres are proved to be in triplicate proportion to their Diameters or Radii; Or might have been join'd as a *Corollary* to *Prop. 17. and 18. Elem. 12.* where the superficies of like Polygons are suppos'd to be inscribed in spheres, having as well the superficies in a duplicate, as the solidity in a triplicate Ratio of the Diameters of the Spheres. These things being premis'd, let us suppose *AE* a Radius or the side of the Sector of a Sphere *EAZ*, to be divided into equal and indefinitely many small pars, and the sector *AEZ* to consist of these spherical superficies *BZ, CZ, DZ, EZ*, it will be evident that all those superficies in the Progression are as the squares of the Radii, that is, as $\overline{AB^2}, \overline{AC^2}, \overline{AD^2}, \overline{AE^2}$, &c. or as the squares of the numbers 1, 2, 3, 4, &c. whence by our second Rule, the sum of all these superficies, that is, the sector *AEZ*, will be $\frac{1}{2}$ of as many superficies equal to the greatest *FZ*, that is, $\frac{1}{2}$ of the greatest *EZ*, drawn into r the number of terms. Whence a sector is equal to a Cylinder whose base is $\frac{1}{2}$ of the greatest or extreme superficies of the sector, and whose altitude is r : Or to a Cone whose base is equal to the superficies of the sector, and its altitude r , which is the last of *Lib. 1.*) but we just now prov'd that a
sector

sector is equal to a Cone whose altitude is r , and base a circle describ'd by the Radius YE, drawn from the vertex of the segment EYZ to the circumference of the base. Wherefore a Cone, whose altitude is, and base equal to, the superficies of the sector, is equal to a Cone of the same altitude, whose base is a circle describ'd by the Radius YE.

And so the superficies of the sector EYZ is equal to a circle describ'd by the Radius YE. Which certainly is the principal Theorem of all those that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; viz. *That the superficies of any segment of a sphere is equal to a circle whose Radius is a right line drawn from the vertex of the segment to the circumference of the bases: And hence, That the superficies of an Hemisphere is double to the base, or equal to two great circles of the sphere.*

For in this Case $YE^2 = AZ^2 + AY^2 = z$
 AE^2 , and consequently a circle described by the Radius YE is equal to two circles describ'd by the Radius AE. Whence also, *the superficies of the whole sphere is quadruple a circle having the same Radius with the sphere, that is, quadruple the greatest circle in the sphere; and equal to a circle whose Radius is the diameter of the sphere.* From hence it follows, *That the superficies of a sphere is equal to the superficies of a Cylinder of the same height and breadth; for the superficies of that Cylinder is quadruple to the base, as we shall shew hereafter. And these are the most noted Theorems of Archimedes.* Nay, from hence all those things follow, which he



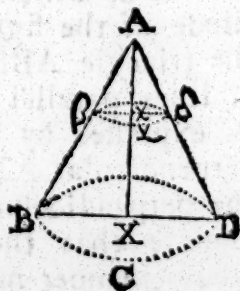
has written concerning the Superficies of Spheres, and their segments. So that from these few and easy Suppositions, I have demonstrated whatever seems to be of any Note in the Books *Of the Sphere and Cylinder*.

I will only add, that after by the method of *Archimedes*, (for I think scarce any other can be invented, besides ours, for finding the solidity) the superficies of segments are found equal to the circle described by the Radii YE; hence it will plainly follow, that the superficies of spheres, and thence of like sectors are in a duplicate ratio of the Radii of the spheres; and consequently from the superficies thus found, the contents of segments, and of whole spheres may be mutually deduced, and that very clearly and expeditiously after this manner. Because in the sector EAZ (fig. Pag. 511.) the superficies BZ, CZ, DZ, EZ, proceed as the squares describ'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, &c. the whole sector will be equal to $\frac{1}{3}$ of as many superficies equal to the greatest EZ, or $EZ \times r$, that is, to a Cylinder whose base is $\frac{1}{3}$ EZ, and altitude r , or to a Cone whose base is EZ, and altitude r . But EZ is supposed equal to a circle whose Radius is YE, wherefore the sector EAZ is equal to a Cone whose base is a circle described by the Radius YZ and altitude r : Which is *Archimedes's* universal Theorem for the contents of Circles. Whence if from this the Cone ZAE standing on the base of the segment EYZ, and having the vertex at the center of the sphere A, be subducted, you'll have that segment EYZ. But when the sector EYZ is a Hemisphere, there will be no such Cone to be subducted; and for that reason a Cylinder whose base is $\frac{2}{3}$ EZ, and altitude r , or

the

the Cone whose base is $2EZ$, and altitude likewise r , will be equal to the whole sphere. But the Superficies of the Hemisphere EZ , is proved to be equal to two of the greatest circles in the sphere, whence the whole sphere is given. This is *Archimedes's* first and principal Theorem, for the content of a sphere; whence 'tis easily deduced, that a sphere is $\frac{2}{3}$ of a circumscrib'd Cylinder, that is, of a Cylinder whose altitude and diameter of its base is equal to the diameter of the sphere.

The Doctrine of our other [*Archimedes*] seems to make against, and subvert the new and celebrated *Method of Indivisibles*, and is press'd to that end by *Tacquet*; for instance, (*Prop. 2. lib. 2. Cylindr.*) For the usual process of that method seems to exhibit the dimension of the superficies of a Cone, (as also of a sphere, and of other Curves) different enough from what our Author and others have demonstrated: As for example, let us suppose $ABCD$ a right Cone, whose Axe is AX , and base BCD , and plane $\beta \chi \delta$ drawn, at pleasure, parallel to the base BCD . And since, as $Diam. BD : Periph. BCD :: Diam. \beta \delta : Periph. \beta \chi \delta$, and so every were it will be (according to the *Method of Indivisibles*, and by 12. 5.) as $Diam. BD$, to $Periph. BCD$, so is the triangle ABD , consisting of those parallel Diameters, to the Conic Superficies $ABCD$, consisting of those Peripheries, i. e. $Diam. BD : Periph. BCD :: AX \times BD : AX \times Periph. BCD$



Whence

$\frac{2}{K}$

$\frac{2}{k}$

AX

$AX \times \text{Periph. BCD}$

_____ will be equal to the Super-

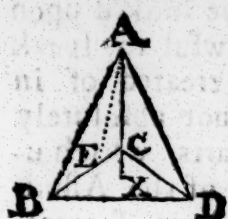
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ficies of the Cone; which is false and contrary to what was demonstrated just now. For we demonstrated that the superficies of the

$AB \times \text{Periph. BCD}$
Cone was _____.

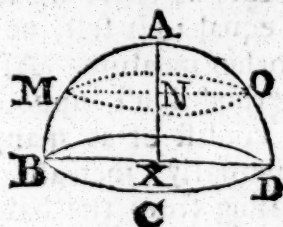
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In answering this Objection, we say, that the *Method of Indivisibles*, in the speculation of Perimeters, and of Curve Surfaces, proceeds otherwise than in the speculation of plane Surfaces and solid Contents. It does indeed suppose that the Area of plane Figures consists as it were of parallel right lines, and the contents of solids of parallel Planes, and that their number may be express'd by the altitude of the Figures: But it by no means supposes, that the Perimeters of plane figures consist of points, or the superficies of solids of lines, the number of which may be express'd by the altitude of the figure. As for example, altho' the triangle ABD (in the last figure) consists of lines parallel to BD, the number of which is expressed by the number of points in the perpendicular AX, that is, by the length of the perpendicular: Yet it would be absurd to suppose that the line AB consists of points, whose number may be express'd by the number of points in a less line AX. For altho' the right line $\beta\zeta$ drawn thro' each infinitely small part of AX, divide AB into as many infinitely small parts, yet those parts are not of the same Denomination or Quality with the parts of AX, but somewhat greater than them; so that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called

points,

points, the parts of AX are to be look'd upon as less than points, if it be lawful to speak so. For the points which are treated of in the *Method of Indivisibles* are not absolutely points, but indefinitely small parts, which usurp the name of points, because of the Affinity. Since therefore points don't admit of greater and less, the name of points is not at the same time to be attributed to the parts of different magnitudes; consequently tho' the number of the greater parts of AB may be express'd by the number of the lesser parts of AX , yet the number of points in AB can no ways be expressed by the number of points in AX , (that is, by the number of parts in AX , equal to the number of parts in AB , which are called points.) The line AB has as many points as there are in it self alone, or another line equal to it self, nor can it be determin'd by any other measure. After the same manner, this method don't suppose the conic Surface $ABCD$ to consist of as many parallel circumferences perpetually increasing from the vertex A , or decreasing from the base BD , as there are points in the Axe AX , but rather of as many thus increasing or decreasing as there are points in the side AB . For in the Revolution of the line AB about the Axis AX , (whereby the superficies of the Cone is generated) every point in the line AB produces a circumference, and consequently more circumferences are produced than the points contained in the Axis AX . Therefore if you would extend the *Method of Indivisibles* to the superficies of solids, and suppose those superficies to consist of parallel lines, you ought not to compute this by the parallel Areas constituting the solid, that is, not to number those Areas by the altitude of the solid, but by other lines agreeable to the condition of each figure. Which lines, in figures that are not irregular, may easily be determin'd: For instance,



stance, in the equilateral Pyramid $ABCD$, whose Axis is AX , supposing that the lateral surface of the pyramid consists of Perimeters of triangles, parallel to the base BCD , these can neither be computed by the altitude AX , nor by the side AB , (for by the former, the thing requir'd, would be wanting of the true Dimension, and by the latter 'twould exceed it) but by the line AE drawn from the vertex A perpendicular to the side BC of the base: The reason of which is, that every plane side of a Pyramid, as ABC , consists of parallel right lines computed by the altitude AE . After the same manner, supposing that the superficies of



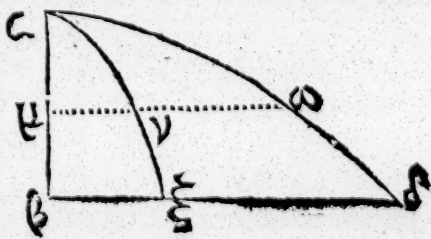
the Hemisphere BAD , consists of peripheries of circles parallel to the base BCD , the number of them is not to be computed by the Axis AX , but by the Quadrantal Arc AB , because that every point of the Arc AD in revolving produces a circumference; and so any superficies, whether plane or curv'd, which is conceived to consist of equidistant right or curv'd lines, is to be computed by a line cutting those equidistant lines perpendicularly. For since those equidistant lines, in this *Method of Indivisibles*, are not consider'd absolutely as lines having an infinitely small breadth, which is the same with the breadth or thickness of the point describing those equidistant lines in their Circumvolution, and since the same equidistant lines divide the line cutting them perpendicularly into parts measuring its breadth, those parts are to be look'd upon as such sort of points, and consequently the number of equidistant lines, or the sum of those breadths is to be com-

computed by the number of points in the line cutting them perpendicularly, that is, by the length of that line, and not by a line of any other length, for that will consist of more or less points.

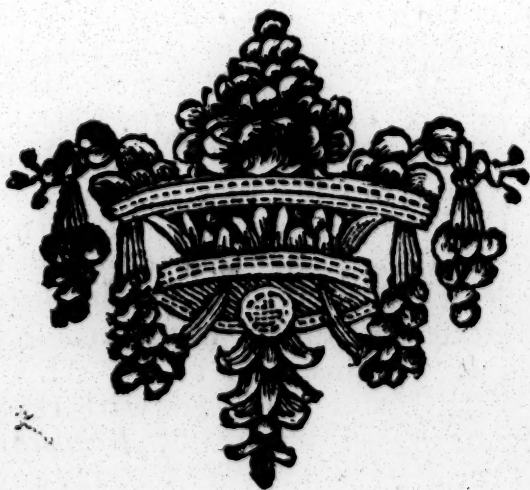
Hence therefore in the speculation of the superficies of solids, the *Method of Indivisibles* is not unuseful, but rather very commodious, provided it be rightly understood, and applied according to the Rule prescrib'd. For by the help of it even these superficies may be found, if so be we have some convenient *Data* presuppos'd, on which the reasoning may be founded: For instance, we might by the help of it, investigate the superficies of a Cone, by reasoning after this manner.

If the superficies of the cone ABC (fig pag. 515.) be divided into innumerable Peripheries of circles $\beta\chi\delta$ parallel to the base BCD, the breadth of those Peripheries taken together, make up the side AB cutting them perpendicularly, and consequently there will be as many Peripheries as there are points in the line AB, that is, their number may be express'd by the number of points in AB, or by its length. Wherefore if you draw perpendiculars equal to the Peripheries to every point of AB, a superficies will be made out of those perpendiculars equal to the superficies of the Cone; but that superficies will be a triangle whose height is AB, and base equal to the greatest Periphery BDC, and so the superficies of the Cone will be $= \frac{1}{2} AB \times \text{Periph. BDC}$, which conclusion agrees with the things laid down and demonstrated by *Archimedes*.

After the same manner, if you take any right line $a\beta$ equal to the quadrantal Arc AB of the Hemisphere (in pag. 518.) and to each of its points μ let the per-



pendiculars $\mu\nu$ be erected equal to the Radii MN of parallel circles MOM passing thro' the corresponding points M of that quadrantal Arc, the greatest of which $\rho\xi$ let be equal to the Radius BX of the base of the Hemisphere: The figure $\alpha\rho\xi$ will contain the Radii of all the circles of whose Peripheries the superficies of the Hemisphere consists. And if the perpendicular $\mu\nu$, $\rho\delta$ be erected equal to the Peripheries MOM, BDB, there will be made the figure $\alpha\rho\delta$ equal to the superficies of the Hemisphere. The dimension of which figure if you can by any means find (as in this case you are to find the Area of the figure $\alpha\rho\xi$) thence you will easily deduce the content of the segment of the sphere, agreeing to what you may gather by any lawful reasoning. Which Observation, I think, will not be unuseful in *Geometry*.





SUPPLEMENT:

Containing some Practical Corollaries
deduced from some of the most ma-
terial Propositions in EUCLIDE,

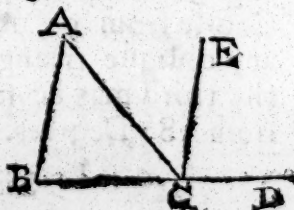
BY

HENRY WILSON.

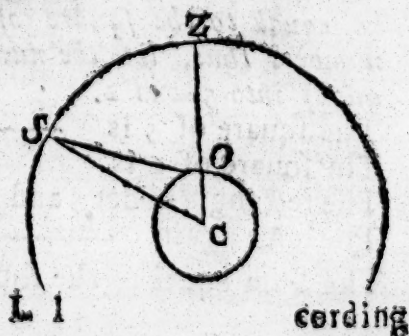
Lib. I. Prop. 32. Part 1.



*I*N any triangle ABC , one side BC being
continued at pleasure, the outward an-
gle ACD shall be equal to the two in-
ward opposite ones
 A and B taken
together.



Hence we see the Reason
and determine the quantity
of the difference between the
true and apparent place of
a Star, commonly called Pa-
rallax; suppose C
the center of the
Earth, O the place
of Observation up-
on the Surface there-
of, S a Star, Z the
Zenith or Vertical
point; the Starac-



cording to its true place, makes with the Vertical point Z, an angle SCZ at the center of the Earth, but at the Eye of the Observer at O it makes the angle SOZ according to its apparent place; but the angle SOZ is equal to the sum of the angles S and C; then the angle S is the difference between C and SOZ, If therefore you subtract the angle C found by Calculation from SOZ found by Observation, the remainder is the angle S the Parallax, but in Practice subtract the Parallax from the apparent Complement of Altitude, the remainder is the true Complement of Altitude or Zenith Distance required.

Lib. 1. Prop. 32. part 2.

In any right-lined triangle the three angles taken together are equal to two right angles, or 180 Degrees; and therefore if one angle be right, the other two taken together will be equal to one right angle.

Hence in Navigation when the Course is given, its Complement (*viz.* the other acute angle) is found by subtracting the angle of the Course from 90 degrees, and *vice versa*; and in any oblique triangle, if two angles be given the third may be found by subtracting their sum from 180 Degrees.

Lib. 2. Prop. 4.

If any line be divided unequally, the square of each part and two rectangles under the parts will be equal to the square of the whole line; or in numbers thus, let the number be 7 divided unequally into 5 and 2.

The square of 5 is 25

The square of 2 is 4

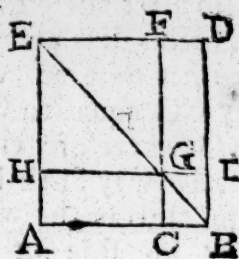
The rectangle under 5 and 2 *viz.* 5 times 2 — 10

The same again 10

The Sum equal to the square of 7, *viz.* 49

Geome.

Geometrically thus, let the line HI be divided unequally in G, then is HF the square of HG, and GB is the square of GI; also GD and GA are two rectangles under HG and GI, and these four fill up the square AD, viz. the square of AB equal to HI which was to be proved.



We have from hence a plain demonstration of the Method made use of in extracting the Root of any square Number; as for instance, it is required to extract the square root of 4624, which let be represented by the square ABDE; the root of 4624 is 68, which let be represented by any side of the said square (which in this Case are always equal) as suppose AB, which is divided in C, the part AC representing the Tens in the root, viz. 60 (which is exprels'd by 6 in the place of Tens) and the latter segment CB the last figure in the root viz. 8. and then the work will stand thus

$$\begin{array}{r}
 4624 \text{ (68)} \\
 128) 1024 \\
 \underline{1024} \\
 0000
 \end{array}$$

The practice as deduced from hence is thus: Having placed and pointed the figures as usual, I find the greatest root in the first Period (46) is 6 which in the place of Tens is 60, which if multiplied by it self is 3600, which takes away the square EFGH whose side or root is HG equal to AC. The next step is to find how much the segment CB is, which is both the side of the square CGIB and the breadth of the Parallelograms, which because they are two, the Root 6 is to be doubled and placed on the left hand of the remainder of the former step with the rest of the figures brought down; then how oft 12 in 102, which is 8 times, set 8 in the root and

L 1 2

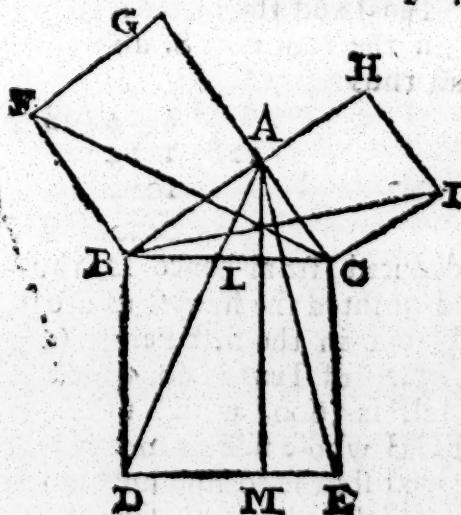
also

also after the 12; multiply 8 by 8 it makes 64 which takes away the square BCGI, then if the former root 60. viz. HG had been multiplied by CG equal to CB it had taken away the Pgr. AG; therefore the former root is doubled, and takes away both Pgrs at once, viz. AG and GD, and thus is the whole square ABDE taken away, and its side or root found to be AC 60 and CB 8 viz. 68.

If there are more figures in the root it is but to renew the Operation in the same manner as before, and the reason may be conceived by dividing the line AB into as many parts as there are figures in the root, and then the Nature and Use of the several Operations will appear very evident.

The square root is of very extensive use in Mathematicks as in that famous Theorem

Lib. 1. Prop. 47.



There it is prov'd that in the Triangle ABC the square BE of the Hypotenuse BC is equal to the square BG of the Leg BA, and the square AI of the Leg AC taken together.

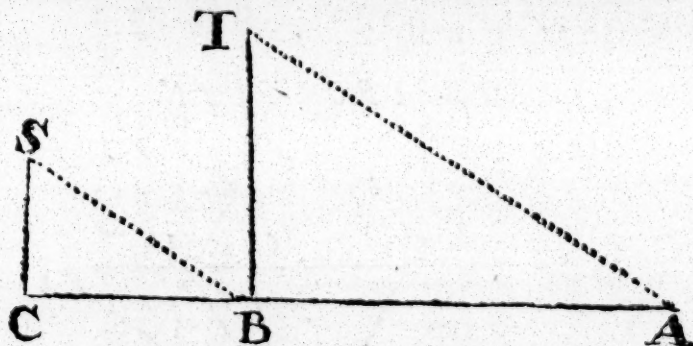
Hence in Navigation if the difference of Latitude and Departure be given the distance may be found, only by the help of the square root; for add the square of the difference of Latitude to the square of the departure, the square root of that sum is the Distance;

stance; but if the Distance be one of the Data, the square root of the Difference of the squares of the parts given is the part or side required.

Lib. 6. Prop. 4.

In equiangular Triangles the sides about the equal Angles are proportional.

Hence, 1. We have a method to measure any inaccessible Height, if upon plain Ground, as suppose a Tower by the Sun at any Altitude, it matters not, nor need the Altitude of the Sun be known at all; for make a mark exactly where the Shadow of the top of the Tower falls, and at the same Moment set up a stick of any given



length, and measure the length of each shadow separately, it will be as the length of the shadow of the stick to that of the Tower, so the length (or height) of the stick to that of the Tower; thus let TB be the Tower, SC the stick, then because the lines SB and TA are parallel, and SC and TB are also parallel, and both perpendicular to the right line CA; the Triangles SCB and TBA are equiangular, and therefore as CB to CS so BA to BT.

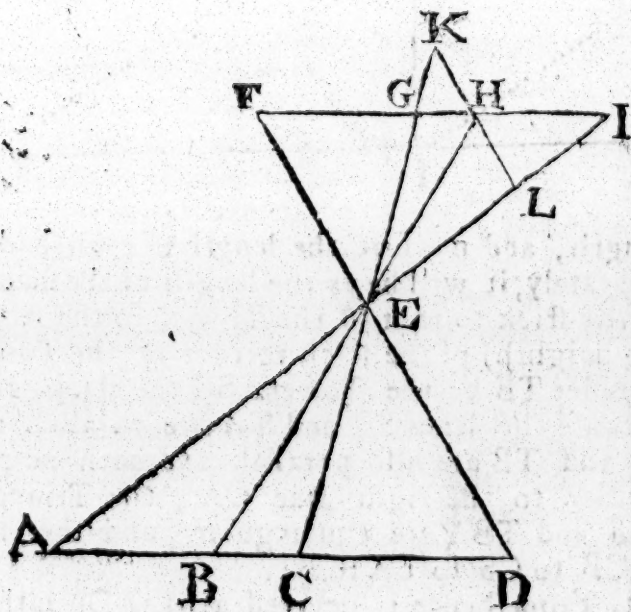
2. From hence is derived all the Operations by Sines, Tangents and Secants, for we rarely meet with a triangle whose side that we make radius is just 1, 000, 000, &c. but having a

COROLLARIES *from*

sufficient Data, assume a triangle in the Table whose radius is an even decimal Number, and angles equal to those in the given triangle, and then as the radius in the table, to the number of parts in the side made radius in the given triangle, so the Sine, Tangent, &c. in the table, to the number of parts contained in the side made Sine, Tangent, &c. in the said triangle, because of the *Proportion of the sides of equiangular triangles*; and these proportions may be altered or inverted at pleasure according to what is given or required, as *Euclide* largely demonstrates in the fifth Book, introduced with proper Definitions of which way of Reasoning the following Problem of my Friend *J. M.* may be a proper Specimen.

To divide a line *AD* so, that as the whole line *AD* to the part *DE* so *AB* to *BE*.

CONSTRUCTION.



Draw *AI* and *DE* to make any angle with the line *AD* and to cut *AI* any, where in *E*; also from

from any point in EI draw LK parallel to EF of length at pleasure, which divide equally in H, making LH equal to HK, and through H draw FI parallel to AD; then from H and K through E, draw HEB and KEC; then shall the line be divided in B and C, so that as AD to DC so AB to BC.

For the triangles FEI and HLI are *a* propor- a 2. 6.
tional, also the triangles FGE and HGK are
similar, *b* because KH is parallel to FE, and *b* *Constr.*
the angles KGH, and FGE are *c* equal. There- c 15. 1.
fore

As FI : FE :: HI : HL.

Therefore as HK (HL) : HI :: FE : FI.

By Permutation, as HK : FE :: HI : FI.

But as HK : FE :: GH : FG.

Destroy the first Terms in the two last Proportions (being the same in both) it will be as
HI : FI :: GH : FG.

By inverse ratio, as FI : HI :: FG : GH.

By Permutation, as FI : FG :: HI : GH.

But the triangles FEG, GEH, HEI are proportional to the triangles DEC, CEB, BEA.

Therefore the line AD is divided in the same proportion as the line FI, viz. AD : DC :: AB : BC. Which was to be done.

From these two excellent Propositions before mentioned, viz. 47. 1. and 4. 6. may be deduced the wonderful Analogy of the Sines, Tangents and Secants, amongst themselves, and from that proportion to find one by the help of another; For

In the Quadrant ACG, let the angle DAB be 35 Degrees, then is AD or AC radius, DB is the Sine of 35, CE is the Tangent, and AE the Secant of 35; also AB equal to FD is the Sine Complement of 35; now if the Sine of 35 DB is given to find the Sine Complement AB, from the square of radius subtract the square of the

radius AC, so radius AG to Tangent GH, the Tangent of the angle GAH, or arch GD the Complement of DC.

By the same Rule the triangles ABD and AGH are equiangular and proportional, that is, as Sine BD to radius AD so radius AG to AH the Secant of the angle GAH or Complement of the angle EAC.

And for the same reason, as Sine Complement of any arch is to radius so is radius to the Secant of that arch.

Thus in those three most useful Proportions we have brought radius to be the two middle terms *viz.*

1. As Sine to radius so radius to Secant Complement.

2. As Sine Complement to radius so radius to Secant.

3. As Tangent to radius, so radius to Tangent Complement.

Hence the square of radius is equal to the rectangle under the Sine and Secant Complement of any arch (and consequently to that under the Sine Complement and Secant) and also to the rectangle under the Tangent and Tangent Complement of any arch, and consequently these rectangles are equal to one another.

There are also other Analogies or proportions of the Sines, Tangents and Secants amongst themselves, which may be deduced from this Proposition, where Radius is not twice express'd, as in the Similar triangles ABD, ACE; for as Radius AD to Sine DB so Secant AE to Tangent CE. Also as Sine Complement AB to Sine BD so Radius AC to Tangent CE; and from hence if we use the natural Sines, Tangents, &c. multiply the second and third Terms together and divide that product by the first (as in the common Rule of Three) the Quotient is the fourth Term required as in the Case above (as

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(as Sine Complement of any Arch to Sine fa Radius to Tangent of that Arch) multiply the natural Sine of any Arch by Radius and divide the product by the Sine Complement of the same, the Quotient is the Tangent of the same Arch, &c.

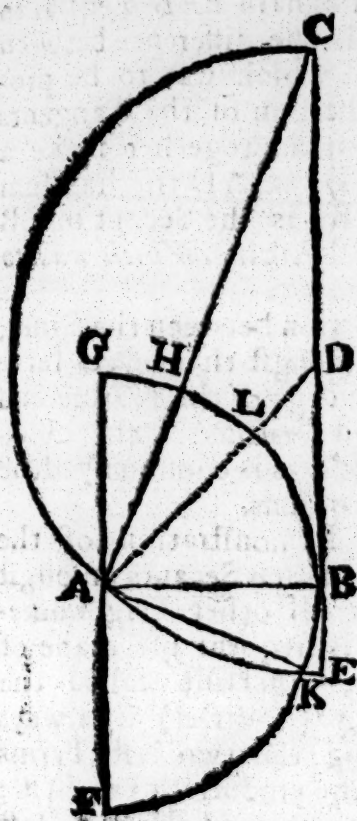
But because in Logarithms Addition performs Multiplication, and Subtraction performs Division, we may hence deduce these useful and necessary Conclusions.

1st, Subtract the Logarithm Sine of any arch from twice radius, the Remainder is the Secant Complement of that arch. 2. (which indeed is the same) Subtract the Sine Complement of any arch from twice radius there will remain the Secant of the same arch. 3. Subtract the Tangent of any arch from twice radius there will remain the Tangent Complement of the same arch. 4. Add the Sine and Secant of any Arch together, and from their sum abate Radius the remainder is the Tangent of the same Arch. 5. Add the Sine Complement of any Arch to the Tangent of the same, and their sum abating Radius is the Sine of the same: And this may be of excellent use in Books and Tables of Logarithmic Sines, Tangents and Secants, for immediately recovering any Sine, Tangent or Secant that is defac'd or obliterated, which could not otherwise be done without a great deal of Trouble.

From hence this may be further observed.

1. That radius is a mean proportional between the Tangent of an Arch, and the Tangent Complement of the same a Arch, and therefore if the arch HK be 90, viz. BH 70, and BK 20, con-

a 13.6.



continue the Tangent line CB to E, then is BC the Tangent of 70, and BE the Tangent of 20. Bisect CE in D, and with the Extent DC or DE draw the semicircle EAC, it shall pass through A the center of the circle GBF, and be a *b* mean proportional between EB and BC.

b13. 64

2. Half the sum of the Tangents BE and BC (*viz.* ED or DC) shall be equal to the Secant of the difference of the said Arches HB and BK, *viz.* LB.

For AD is equal to DC because both semi-diameters of the

same semicircle EAC, and is the Secant of the Arch LB, now that the arch LB is the difference between HB and BK is thus proved.

The angle ABC is *c* right, because BC is *c* *constr.*
 a Tangent line, also EAC is right, because HK
 its measure is a *d* Quadrant, therefore BAC and *d* *constr.*
 BCA together is equal to *e* one right angle, also *e* 32. 1.
 EAD together with DAC are equal to one
 right angle, but DAC is equal to DCA *f* be- *f* 18. 1.
 cause their opposite sides are equal, therefore
 DCA (BCA) with EAD are equal to one right
 angle, but DCA with BAC are also equal to one
 right angle; take away DCA common to both,
 there remains EAD *g* equal to BAC; again from the *g* 3. Ax. 1.
 triangles BAC and EAD take away BAD com-
 mon

h 3. ax. 1. mon to both, there will remain EAB $\hat{=}$ equal to DAC; therefore DAB is the difference between the Arches HB and BK which was to be proved; and hence half the sum of the Tangents of any two Arches which together make a Quadrant, viz. $ED = DE = AD$ the half sum of the Tangents BE + BC is the Secant of LB the difference of the said Arches. Which was to be demonstrated.

This wonderful Proportion between the Sines, Tangents and Secants amongst themselves serve to give a true understanding of the Nature and Application of them, and Reasons of the changing Secants for Sines, &c. as is commonly done in Trigonometrical Operations.

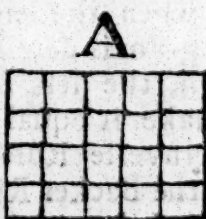
But whereas in this Demonstration of the Analogy of Sines, Tangents and Secants amongst themselves, and also in all other Trigonometrical Operations there is frequent use made of the Rule of Proportion, otherwise called the Rule of Three, which (if direct) is always perform'd by multiplying the two last Terms together and dividing the product by the first; or if done by Logarithms, to add the two last Terms together, and from their sum subtract the first, &c. it may not be amiss to give a demonstrative Reason of these Methods of Operation, which may be done from

Lib. 6. Prop. 16.

If four right lines be proportional, the rectangle comprehended under the Extremes, or first and last Terms, is equal to the Rectangle comprehended under the means or two middle Terms [See the Demonstration in Lib. 6. Prop. 16.] as if we assume four Magnitudes A, B, C, D, proportional, as A to B so C to D, the rectangle under B and C is equal to that under A and D, viz. the Product of C and B multiplied together is equal to that of A and D so multiplied.

plied. Now in the common Rule of Three, there are always three Numbers given to find a fourth, which shall have the same Proportion to the third as the second hath to the first, it will follow that the number required is to be such a number as when multiplied by the first Term the Product shall be equal to that of the second multiplied by the third; but as Multiplication and Division undo each other, the way to know what number it is which multiplied by any given Number the product shall be equal to another given Number, divide the said Number by the Number to be multiplied, the Quotient is the Number required; as for instance, I would know what Number multiplied by two is equal to five Multiplied by four, five Multiplied by four the Product is twenty; divide twenty by two the Product is ten; so ten multiplied by two is equal to five multiplied by four.

Or in the Parallelograms annex'd, that mark'd A hath one side five the other four parts which by lines continued through divides it into twenty (*viz.* five times four) small squares, the other mark'd B whose



breadth is but two, and it is required to know how long it must be to contain likewise twenty squares, or as it is but two squares in breadth how often that two must be repeated to make twenty, which is ten times, as appears by the figure, which proves that ten in length, and 2 in breadth is a rectangle equal to 5 in length and four in breadth; and as in four Lines or Magnitudes proportional as before, the rectangle under the

the means are proved equal to that under the Extrems, and the unknown side of one Parallelogram is found by dividing the other Parallelogram by the known side; from hence ariseth the Reason of multiplying the second and third Terms together in the Rule of Three, and dividing that product by the first to find the fourth Term required which answers the Question.

The Rule of Three Reverse will admit of a like Demonstration tho' of a quite contrary Nature to the former; for whereas in the Rule of Three direct, *more* requires *more* and *less* requires *less*; in this, *more* requires *less*, and *less* requires *more*; or in the former Parallelograms when the breadth of that mark'd B is determined to be two, the more the length and breadth of that mark'd A, the more must be the length of that mark'd B to make it equal to the other; but in the Inverse Rule of Three *more* requires *less*, &c. viz. when the length and breadth of one Parallelogram is determined, then the greater the length, the less breadth is required in another to make it equal to the former. The nature of the Inverse Rule of Proportion (commonly called the Backer Rule of Three) is best demonstrated, *Lib. 6. prop. 14.* where it is proved that equal Parallelograms, having their angles equal, have the sides about the equal angles reciprocal; and those Parallelograms, which have equal angles and their sides reciprocal, are equal, *see Lib. 6. Prop. 14.* where you have it demonstrated: As in the equal Parallelograms A and B before mentioned, if applied according to that Demonstration it is made evident; for suppose there is given the rectangle A whole sides are four and five, and the lesser side (two) of the rectangle B, then as the lesser side (two) of the rectangle B to the lesser side four of the rectangle A, so reciprocally the greater side of the rectangle A, viz. five, to the
greater

greater side of the rectangle B which is ten, so that Inverse Proportion is perform'd by Multiplication and Division as direct, and demonstrated the same way if a due care be taken to consider which two Terms to take together to make the rectangles equal, and to proceed to multiply and divide accordingly.

And hence we may conclude that Geometrical Proportion is, when, in four Numbers the first contains the second as oft as the third contains the fourth, or the second contains the first as oft as the fourth the third, &c. or in three Numbers, when the first contains the second as oft as the second contains the third. Let the four proportional Numbers be $3 : 6 :: 8 : 16$. viz. 3 is contained in 6 as often as 8 is in 16, &c.

Or when the Proportion is in three Numbers, viz. at 4 to 6 so 6 to 9, that is, 4 is as often contained in 6 as 6 is in 9, and hence 6 is a mean Proportional between 4 and 9, because the square of 6 is equal to the rectangle under 4 and 9, viz. 36, and the same if the Antecedent be greater than the Consequent, as $16 : 4 :: 20 : 5$; or $18 : 12 :: 12 : 8$. And for Proportions Alternate, Compounded, Divided, &c. it is explained in the Definitions in the fifth Book.

Multiplication and Division are but Branches of the same Proportion, only with this Difference, that whereas there is in the Rule of Three three Numbers given to find a fourth, there seems in this Case to be but two Numbers (suppose Multiplicand and Multiplier) given to find a third, viz. the Product; or Dividend and Divisor to find the Quotient: For in these Cases, Unity or one is always the first Term, and the Proportion is as 1 to the Multiplier so the Multiplicand to the Product, so that as in the Rule of Proportion even so in this the design of Multiplication is, having the length and breadth of a Parallelogram

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rallelogram given to find the length of another Parallelogram equal to it, whose breadth shall be one or Unity, and hence Multiplication may properly be said to serve instead of many Additions.

And as Multiplication reduceth a Quantity from a rectangle whose two sides are given to another equal to it, one of whose sides are Unity or 1, Division performs the quite contrary; and hence as in Multiplication it is as one or Unity to the Multiplier, so the Multiplicand to the product, or consequently (as before mentioned) the rectangle under the Multiplicand and Multiplier is equal to the rectangle under 1 and the product; but in Division, as the Divisor to 1 so the Dividend to the Quotient, and from hence ariseth the Reason in Multiplying by Logarithms which is thus perform'd; Add the Logarithm of the Multiplicand to that of the Multiplier, the sum (because the Logarithm of 1 which should be abated is nothing) is the Logarithm of the Product, and in Division subtract the Logarithm of the Divisor from that of the Dividend, the Remainder shall be the Logarithm of the Quotient, and hence again to Multiply any Number by its self is but to double its Logarithm, consequently to extract the square root of any Number is but to take half its Logarithm, which shall be the Logarithm of the square root required, and again as the Cube of any Number is that Number multiplied by it self, and that product again by the same Number, consequently adding three times the Logarithm of any Number or multiplying it by three, the Sum or Product shall be the Logarithm of the said Cube, and hence if the Cube root of any number be required, divide the Logarithm of the Number by three, the Quotient shall be the Logarithm of the Cube root required.

F I N I S.



